

COHOMOLOGY THEORIES ON LOCALLY CONFORMALLY SYMPLECTIC MANIFOLDS

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ABSTRACT. In this note we introduce primitive cohomology groups of locally conformally symplectic manifolds (M^{2n}, ω, θ) . We study the relation between the primitive cohomology groups and the Lichnerowicz-Novikov cohomology groups of (M^{2n}, ω, θ) , using and extending the technique of spectral sequences developed by Di Pietro and Vinogradov for symplectic manifolds. We discuss related results by many peoples, e.g. Bouche, Lychagin, Rumin, Tseng-Yau, in light of our spectral sequences. We calculate the primitive cohomology groups of a $(2n + 2)$ -dimensional locally conformally symplectic nilmanifold as well as those of a l.c.s. solvmanifold. We show that the l.c.s. solvmanifold is a mapping torus of a contactomorphism, which is not isotopic to the identity.

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CONTENTS

1. Introduction	1
2. Basic operators on a l.c.s. manifold	3
3. Primitive forms and primitive (co)homologies	6
4. Spectral sequences on a l.c.s. manifold	14
5. The stabilization of the spectral sequences	23
6. Kähler manifolds	36
7. Examples	37
Acknowledgement	42
References	42

1. INTRODUCTION

A differentiable manifold (M^{2n}, ω, θ) provided with a non-degenerate 2-form ω and a closed 1-form θ is called a locally conformally symplectic (l.c.s.) manifold, if $d\omega = -\omega \wedge \theta$, $d\theta = 0$. The 1-form θ is called the Lee form of ω .

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The class of l.c.s. manifolds has attracted strong interests among geometers in recent years. For instance, Vaisman showed that l.c.s. manifolds may be viewed as phase spaces for a natural generalization of Hamiltonian dynamics [30]. Bande and Kotschick showed that a pair composed of a contact manifold and a contactomorphism is naturally associated with a l.c.s. manifold [3] (see also Proposition 7.3 and Proposition 7.4 below). Furthermore, l.c.s. manifolds together with contact manifolds are the only transitive Jacobi manifolds [22, Remark 2.10]. It is also worth mentioning that locally conformally Kähler manifolds, a natural subclass of l.c.s. manifolds, are actively studied in complex geometry, e.g. see [15], [31].

Note that a l.c.s. manifold is locally conformally equivalent to a symplectic manifold, i.e. locally $\theta = df$ and $\omega = e^{-f}\omega_0$, $d\omega_0 = 0$. By the Darboux theorem all symplectic manifolds of the same dimension are locally equivalent. Hence symplectic manifolds have only global invariants, and cohomological invariants are most natural among them. First (co)homological symplectic invariants were proposed in works by Gromov and Floer then followed by works by McDuff, Hofer and Salamon, Fukaya and Ono, Ruan, Tian, Witten and many others, including the first author of this note. This approach was based on the use of the theory of elliptic differential operators with purpose to make regular certain Morse (co)homology theory or the intersection theory on the infinite dimensional loop space on a symplectic manifold M^{2n} , or on the space of holomorphic curves on M^{2n} . This elliptic (co)homology theory has huge success, but the computational part of the theory is quite complicated. Almost at the same time, a “linear” symplectic cohomology theory has been developed, beginning with the paper by Bouche [4], and then by other peoples (see [11], [7], [29]). This theory is mostly motivated by analogues in Kähler geometry, the Dolbeault theory, and the cohomology theory for differential equations developed by Vinogradov and his school.

This linear symplectic cohomology theory has not yet drawn as much attention as it, to our opinion, should have. This is, probably, due to the fact that its potentially important applications are still in a phase of elaboration. The computational part of the linear theory seems to be not so complicated as in the elliptic theory, and this is an advantage of its.

In our note we further develop the linear symplectic cohomology theory and extend it to l.c.s. manifolds. This is possible due to the validity of the Lefschetz decomposition for these manifolds. The main tool is the spectral sequence developed by Di Pietro and Vinogradov for symplectic manifolds, which has been now adapted and developed further for l.c.s. manifolds. We obtain new results and applications even for symplectic manifolds. In particular we unify various isolated results of the linear symplectic cohomology theory.

The structure of this note is as follows. In section 2 we introduce important linear operators on l.c.s. manifolds and study their properties. The Lefschetz filtration on the space $\Omega^*(M^{2n})$ of a l.c.s. manifold (M^{2n}, ω, θ)

is discussed in section 3 together with differential operators respecting this filtration. Then we use this filtration to construct primitive (co)homology groups for (M^{2n}, ω, θ) (Definition 3.8). Some simple properties of these groups are fixed in Proposition 3.9, Proposition 3.10 and Proposition 3.13, and their relations with previously proposed constructions are discussed (Remark 3.15). The spectral sequence associated with the Lefschetz filtration is studied in section 4. In particular, its E_1 -term is compared with the primitive (co)homological groups (Lemma 4.1) and the conformal invariance of this term is proven (Theorem 4.6). In section 5 we find some cohomological conditions on (M^{2n}, ω, θ) under which this spectral sequence stabilizes at the E_t -term (Theorems 5.2, 5.8, 5.13). The last of these theorems gives an answer to the Tseng-Yau question on relations between the primitive cohomology and the de Rham cohomology of a compact symplectic manifold. In section 6 we specialize the previous theory to Kähler manifolds and prove that for Kähler manifolds the spectral sequence stabilizes already at its first term (Theorem 6.2). In section 7 we compute the primitive cohomology groups of a compact $(2n + 2)$ -dimensional l.c.s. nilmanifold and a compact 4-dimensional l.c.s. solvmanifold (Propositions 7.1, 7.2). We study some properties of primitive cohomology groups of l.c.s. manifolds associated with a co-orientation preserving contactomorphism (Proposition 7.4). In particular, we show that the compact l.c.s. solvmanifold is associated with a non-trivial co-orientation preserving contactomorphism (Theorem 7.6).

The cohomological theory developed in this note and its analogues have a much wider area of applications. For instance, it may be naturally adopted to the class of Poisson symplectic stratified spaces introduced in [17], since these singular symplectic spaces also enjoy the Lefschetz decomposition.

This project was started as a joint work of us with Alexandre Vinogradov based on H.V.L. preliminary results on l.c.s. manifolds. Alexandre Vinogradov has suggested us to extend the results to a slightly larger category of twisted symplectic manifolds. He made considerable contributions to improve the original text written by H.V.L., which we appreciate very much. Eventually we have noticed that our viewpoints are so different, so we decide to write the subject separately: in this paper we deal only with l.c.s. manifolds and Alexandre Vinogradov will deal with the extension to twisted symplectic manifolds.

2. BASIC OPERATORS ON A L.C.S. MANIFOLD

In this section we introduce and study basic linear differential operators acting on differential forms on a l.c.s. manifold (M^{2n}, ω, θ) .

The first operator we need is the Lichnerowicz deformed differential $d_\theta : \Omega^*(M^{2n}) \rightarrow \Omega^*(M^{2n})$,

$$(2.1) \quad d_\theta(\alpha) := d\alpha + \theta \wedge \alpha.$$

Clearly $d_\theta^2 = 0$ and $d_\theta(\omega) = 0$. The resulting Lichnerowicz cohomology groups (also called the Novikov cohomology groups) are important conformal invariants of l.c.s. manifolds.

Recall that two l.c.s. forms ω and ω' on M^{2n} are *conformally equivalent*, if $\omega' = \pm(e^f)\omega$ for some $f \in C^\infty(M^{2n})$. In this case the corresponding Lee forms θ and θ' are cohomologous: $\theta' = \theta \mp df$, hence d_θ and $d_{\theta'}$ are *gauge equivalent*:

$$d_{\theta'}(\alpha) = (d_\theta \mp df \wedge) \alpha = e^{\pm f} d(e^{\mp f} \alpha).$$

It follows that $H^*(\Omega^*(M^{2n}), d_\theta)$ and $H^*(\Omega^*(M^{2n}), d_{\theta'})$ are isomorphic. The isomorphism $I_f : H^*(\Omega^*(M^{2n}), d_\theta) \rightarrow H^*(\Omega^*(M^{2n}), d_{\theta'})$ is given by the conformal transformation $[\alpha] \mapsto [\pm e^f \alpha]$.

Note that d_θ does not satisfy the Leibniz property, unless $\theta = 0$, since

$$(2.2) \quad d_\theta(\alpha \wedge \beta) = d_\theta \alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d_\theta \beta = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d_\theta \beta.$$

Thus the cohomology group $H^*(\Omega^*(M^{2n}), d_\theta)$ does not have a ring structure, unless $\theta = 0$. The formula (2.2) also implies that $H^*(\Omega^*(M^{2n}), d_\theta)$ is a $H^*(M, \mathbb{R})$ -module.

Now let us consider the next basic linear operator

$$(2.3) \quad L : \Omega^*(M^{2n}) \rightarrow \Omega^*(M^{2n}), \quad \alpha \mapsto \omega \wedge \alpha.$$

Substituting $\alpha := \omega$ in (2.2) we get a nice relation between d , L and d_θ

$$(2.4) \quad d_\theta L = Ld.$$

The identity (2.4) suggests us to consider a family of operators $d_{k\theta}$, which we abbreviate as d_k if no misunderstanding occurs. We get immediately from (2.4)

$$(2.5) \quad d_k L^p = L^p d_{k-p}.$$

The following Lemma is a generalization of (2.2) and it plays an important role in our study of the spectral sequences introduced in later sections. It is obtained by straightforward calculations, so we omit its proof.

Lemma 2.1. *For any $\alpha, \beta \in \Omega^*(M^{2n})$ we have*

$$(2.6) \quad d_{k+l}(\alpha \wedge \beta) = d_k \alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d_l \beta.$$

Consequently

$$(2.7) \quad d_k \alpha \wedge d_l \beta = d_{k+l}(\alpha \wedge d_l \beta).$$

Formula (2.6) yields the induced map $H^*(\Omega^*(M^{2n}), d_k) \times H^*(\Omega^*(M^{2n}), d_l) \rightarrow H^*(\Omega^*(M^{2n}), d_{k+l})$.

Denote by G_ω the section of the bundle $\Lambda^2 TM^{2n}$ such that for all $x \in M^{2n}$ the linear map $i_{G_\omega(x)} : T_x^* M^{2n} \rightarrow T_x M^{2n}$, $V \mapsto i_V(G_\omega(x))$, is the inverse of the map $I_\omega : T_x M^{2n} \rightarrow T_x^* M^{2n}$, $V \mapsto i_V \omega$. Clearly G_ω defines a bilinear pairing: $T^* M^{2n} \times T^* M^{2n} \rightarrow C^\infty(M^{2n})$. Denote by $\Lambda^p G_\omega$ the associated pairing: $\Lambda^p(T^* M) \times \Lambda^p(T^* M) \rightarrow C^\infty(M^{2n})$. The l.c.s. form ω and the

associated bi-vector field G_ω define a *l.c.s. star operator* $*_\omega : \Omega^p(M^{2n}) \rightarrow \Omega^{2n-p}(M^{2n})$ as follows [5, §2.1].

$$(2.8) \quad *_\omega : \Omega^p(M^{2n}) \rightarrow \Omega^{2n-p}(M^{2n}), \beta \wedge *_\omega \alpha := \Lambda^p G_\omega(\beta, \alpha) \wedge \frac{\omega^n}{n!},$$

for all $\alpha, \beta \in \Omega^p(M^{2n})$. Using [5, Lemma 2.1.2] we get easily

$$(2.9) \quad *_\omega^2 = Id.$$

We define the *l.c.s. adjoint* L^* of L and the *l.c.s. adjoint* $(d_k)_\omega^*$ of d_k with respect to the l.c.s. form ω as follows

$$(2.10) \quad L^* : \Omega^p(M^{2n}) \rightarrow \Omega^{p-2}(M^{2n}), \alpha^p \mapsto - *_\omega L *_\omega \alpha^p.$$

$$(2.11) \quad (d_k)_\omega^* : \Omega^p(M^{2n}) \rightarrow \Omega^{p-1}(M^{2n}), \alpha^p \mapsto (-1)^p *_\omega d_{n+k-p} *_\omega (\alpha^p).$$

For symplectic manifolds our definition of $(d_k)_\omega^*$ agrees with the one in [35, §1], it is different from the one in [5, Theorem 2.2.1] by sign (-1).

A section g of the bundle $S^2 T^* M^{2n}$ is called a *compatible metric*, if there is an almost complex structure J on M^{2n} such that $g(X, Y) = \omega(X, JY)$. In this case J is called a *compatible almost complex structure*. Recall that the Hodge operator $*_g$ is defined as follows

$$(2.12) \quad *_g : \Omega^p(M^{2n}) \rightarrow \Omega^{2n-p}(M^{2n}), \beta \wedge *_g \alpha := \Lambda^p G_g(\beta, \alpha) \wedge \frac{\omega^n}{n!},$$

where $G_g \in \Gamma(S^2 T^* M^{2n})$ is the “inverse of g ”, i.e. it is defined in the same way as we define G_ω above: for all $x \in M^{2n}$ the linear map $i_{G_g(x)} : T_x^* M^{2n} \rightarrow T_x M$, $V \mapsto i_V(G_g(x))$, is the inverse of the map $I_g : T_x M \rightarrow T_x^* M$, $V \mapsto i_V(g)$. We also denote by $\Lambda^p G_g$ the associated pairing: $\Lambda^p(T^* M) \times \Lambda^p(T^* M) \rightarrow C^\infty(M^{2n})$ induced by G_g , (see also [5, p.105] for comparing $\Lambda^p G_\omega$ with $\Lambda^p G_g$).

Using [32, Lemma 5.5] we get easily

$$(2.13) \quad *_g^2(\alpha^p) = (-1)^p \alpha^p \text{ for } \alpha^p \in \Omega^p(M^{2n}).$$

Lemma 2.2. 1. *The space of metrics compatible with a given l.c.s. form $\omega \in \Omega^2(M^{2n})$ is contractible.*

2. (cf. [32, chapter II, 6.2.1]) *In the presence of a compatible metric g on M^{2n} we have*

$$(2.14) \quad L^* = \Lambda$$

where $\Lambda = (*_g)^{-1} L *_g$ is the adjoint of L with respect to the metric g .

Proof. 1. The proof for the first assertion goes in the same way as for the case of symplectic manifolds, so we omit its proof.

2. The second assertion of Lemma 2.2 is a simple consequence of the following

Lemma 2.3. [5, Theorem 2.4] *Assume that (M^{2n}, J, g) is an almost Hermitian manifold and ω is the associated almost symplectic form. For $\alpha \in \Omega^{p,q}(M^{2n})$ we have*

$$*_\omega(\alpha) = \sqrt{-1}^{p-q} *_g(\alpha).$$

Here we extend $*_\omega$ and $*_g$ \mathbb{C} -linearly on $\Omega^*(M^{2n}) \otimes \mathbb{C}$.

This completes the proof of Lemma 2.2. \square

Let $\pi_k : \Omega^*(M^{2n}) \rightarrow \Omega^k(M^{2n})$ be the projection. Denote $\sum_{i=0}^{2n} (n-i)\pi_i$ by A . Using well-known identities in Kähler geometry for (Λ, L, A) , see e.g. [33, (IV), chapter I], [13, p.121], [32, Lemma 6.19], Lemma 2.2 implies immediately the following

Corollary 2.4. (cf. [21, §1], [35, Corollary 1.6]) *On any l.c.s. manifold (M^{2n}, ω, θ) we have*

$$(2.15) \quad L^* = i(G_\omega),$$

$$(2.16) \quad [L^*, L] = A, [A, L] = -2L, [A, L^*] = 2L^*.$$

The relation in (2.16) shows that (L^*, L, A) forms a \mathfrak{sl}_2 -triple, which has many important consequences for twisted symplectic manifolds.

Proposition 2.5. *The following commutation relation hold*

$$(2.17) \quad L^*(d_k)_\omega^* = (d_{k-1})_\omega^* L^*.$$

Proof. Clearly (2.17) is obtained from (2.5) by applying the LHS and RHS of (2.5) the l.c.s. star operator on the left and on the right, taking into account (2.9). \square

3. PRIMITIVE FORMS AND PRIMITIVE (CO)HOMOLOGIES

In this section we introduce the notions of primitive forms and coeffective forms on a l.c.s. manifold (M^{2n}, ω, θ) using the linear operators L and L^* defined in the previous section. As in the symplectic case we get a Lefschetz decomposition of the space $\Omega^*(M^{2n})$ induced by primitive forms and coeffective forms together with various linear differential operators respecting this decomposition as well as an associated filtration (Propositions 3.5 and 3.6). The natural splitting of the introduced differential operators according to the Lefschetz decomposition leads to new (co)homology groups of (M^{2n}, ω, θ) (Definition 3.8). In Propositions 3.9, 3.10, 3.13 we fix simple properties of these new (co)homology groups. At the end of this section we compare our construction with related constructions in [21], [4], [25], [10], [35], [7], [8].

Definition 3.1. ([4], [35], cf. [33], [13]) An element $\alpha \in \Lambda^k T_x^* M^{2n}$, $0 \leq k \leq n$, is called *primitive* (or *effective*), if $L^{n-k+1}\alpha = 0$. An element $\alpha \in \Lambda^k T_x^* M^{2n}$, $n+1 \leq k \leq 2n$, is called *primitive*, if $\alpha = 0$. An element $\beta \in \Lambda^k T_x^* M^{2n}$ is called *coeffective*, if $L\beta = 0$.

Remark 3.2. 1. Wells in [34] refers to Lefschetz [18] and Weil [33] for the terminology “Lefschetz decomposition” and “primitive forms”. Many mathematicians prefer “Lepage decomposition” and “effective forms” following Lepage in [19].

2. Clearly the notion of a primitive form as well as the notion of coeffective form depends only on the conformal class of a l.c.s. form ω .

The relation (2.16) between linear operators L, L^* and A leads to Lemma 3.3 below characterizing primitive forms and coeffective forms. The resulting Lefschetz decomposition of the space $\Lambda T^* M^{2n}$ is a direct consequence of the $\mathfrak{sl}(2)$ -module theory. Various variants of Lemma 3.3 for symplectic manifolds appeared in many works, beginning possibly with a paper by Lepage [19], with later applications in Kähler geometry [33], [13], [32], in a theory of second-order differential equations [21], in symplectic geometry [4], [35], etc..

We denote by $P_x^k(M^{2n})$ the set of primitive elements in $\Lambda^k T_x^* M^{2n}$.

Lemma 3.3. 1. An element $\alpha \in \Lambda^k T_x^* M^{2n}$, is primitive, if and only if $L^* \alpha = 0$.

2. An element $\beta \in \Lambda^k T_x^* M^{2n}$ is coeffective, if and only if $*_{\omega} \beta$ is primitive.

3. We have the following Lefschetz decomposition for $n \geq k \geq 0$

$$(3.1) \quad \Lambda^{n-k} T_x^* M^{2n} = P_x^{n-k}(M^{2n}) \oplus L P_x^{n-k-2}(M^{2n}) \oplus L^2 P_x^{n-k-4}(M^{2n}) \oplus \dots,$$

$$(3.2) \quad \Lambda^{n+k} T_x^* M^{2n} = L^k P_x^{n-k}(M^{2n}) \oplus L^{k+1} P_x^{n-k-2}(M^{2n}) \oplus \dots$$

From Lemma 3.3 we get immediately

Corollary 3.4. 1. $L^k : \Lambda^{n-k} T_x^* M^{2n} \rightarrow \Lambda^{n+k} T_x^* M^{2n}$ is an isomorphism, for $0 \leq k \leq n$.

2. $L : \Lambda^{n-k-2} T_x^* M^{2n} \rightarrow \Lambda^{n-k} T_x^* M^{2n}$ is injective, for $k = -1, 0, 1, \dots, n-2$.

It is useful to introduce the following notations. Denote by $P^{n-k} M^{2n}$ the subbundle in $\Lambda T^* M^{2n}$ whose fiber is $P_x^{n-k}(M^{2n})$. Let $\mathcal{P}^{n-k}(M^{2n}) \subset \Omega^{n-k}(M^{2n})$ be the space of all smooth $(n-k)$ -forms with values in $P^{n-k} M^{2n}$. Elements of $\mathcal{P}^{n-k}(M^{2n})$ are called *primitive $(n-k)$ -forms*. Let us set (cf. [29])

$$(3.3) \quad \mathcal{L}^{s,r} := L^s \mathcal{P}^r \text{ for } 0 \leq s, r \leq n.$$

Put $\mathcal{P}^*(M^{2n}) := \bigoplus_r \mathcal{P}^r(M^{2n})$. Then Lemma 3.3 yields the following decompositions, which we call *the first and second Lefschetz decompositions*

$$(3.4) \quad \Omega^*(M^{2n}) = \mathcal{P}^*(M^{2n}) \oplus L \Omega^*(M^{2n}) = \bigoplus_{0 \leq 2s+r \leq 2n} \mathcal{L}^{s,r}.$$

Now we consider the interplay between the Lefschetz decompositions and the linear differential operators introduced in the previous section. Since $d_p(\omega \wedge \alpha^k) = \omega \wedge d_{p-1} \alpha^k$ for $\alpha^k \in \Omega^k(M^{2n})$, we have a natural differential

ideal $\mathcal{J}_p \subset (\Omega^*(M^{2n}), d_p)$ generated by $L\Omega^*(M^{2n})$. This differential ideal defines the following filtration on $K_p^* := \Omega^*(M^{2n})$.

$$(3.5) \quad \begin{aligned} F^0 K_p^* &:= K_p^* \supset F^1 K_p^* := LK_{p-1}^* \supset \cdots \\ &\supset F^k K_p^* := L^k K_p^* \supset \cdots \supset F^{n+1} K_p^* = \{0\}. \end{aligned}$$

Proposition 3.5. 1. The subset $F^k K_p^*$ is stable with respect to d_p for all k and p .

2. For any $\gamma \in \Omega^1(M^{2n})$ we have

$$\gamma \wedge \mathcal{L}^{0,n-k} \subset \mathcal{L}^{0,n-k+1} \oplus \mathcal{L}^{1,n-k-1}.$$

Proof. 1. The first assertion of Proposition 3.5 follows from the identity $d_p(\omega^k \wedge \phi) = \omega^k \wedge d_{p-k}\phi$ for $\phi \in \Omega^*(M^{2n})$.

2. Assume that $\alpha \in \mathcal{P}^{n-k}(M^{2n}) = \mathcal{L}^{0,n-k}$. Then $L^{k+1}(\gamma \wedge \alpha) = \gamma \wedge L^{k+1}(\alpha) = 0$. Taking into account the decomposition of $\gamma \wedge \alpha$ according to the second Lefschetz decomposition we get the second assertion of Proposition 3.5 immediately. \square

We observe that the decompositions (3.1), (3.2) and (3.4) are compatible with the filtration (3.5) in the following sense. For any $p \geq 0$ we have

$$(3.6) \quad F^p K_r^* \cap \Omega^{n-k}(M^{2n}) = \bigoplus_{i=0}^{\lfloor \frac{n-k}{2} \rfloor - p} \mathcal{L}^{p+i, n-k-2p-2i} \text{ if } k \geq 0 \text{ and } (n-k)/2 \geq p,$$

$$(3.7) \quad F^p K_r^* \cap \Omega^{n-k}(M^{2n}) = 0 \text{ if } k \geq 0 \text{ and } (n-k)/2 < p,$$

$$(3.8) \quad F^p K_r^* \cap \Omega^{n+k}(M^{2n}) = 0 \text{ if } k > 0 \text{ and } (n+k)/2 < p,$$

$$(3.9) \quad F^p K_r^* \cap \Omega^{n+k}(M^{2n}) = \bigoplus_{i=0}^{\lfloor \frac{n+k}{2} \rfloor - p} \mathcal{L}^{p+i, n-k-2p-2i} \text{ if } k > 0 \text{ and } (n+k)/2 \geq p.$$

Proposition 3.6. The following inclusions hold

$$(3.10) \quad d_r \mathcal{L}^{p,q-p} \subset \mathcal{L}^{p,q-p+1} \oplus \mathcal{L}^{p+1,q-p-1}.$$

$$(3.11) \quad (d_r)_\omega^* \mathcal{P}^{n-k}(M^{2n}) \subset \mathcal{P}^{n-k-1}(M^{2n}).$$

Proof. Let $\beta \in \mathcal{P}^q(M^{2n}) = \mathcal{L}^{0,q}$, so $L^{n-q+1}\beta = 0$. By (2.5) we get

$$(3.12) \quad L^{n-q+1}d_r\beta = d_{r+n-q+1}L^{n-q+1}\beta = 0.$$

Using (3.1) and (3.2) we get $d_r\beta \in \mathcal{P}^{q+1}(M^{2n}) + L\mathcal{P}^{q-1}(M^{2n})$. This proves the inclusion (3.10) of Proposition 3.6 for $p = 0$. The inclusion (3.10) for $p \neq 0$ follows from the particular case $p = 0$ and the identity $d_r L^p = L^p d_{r-p}$.

Assume that $\beta \in \mathcal{P}^{n-k}(M^{2n})$. Taking into account (2.17) we get

$$L^*(d_r)_\omega^* \beta = (d_{r-1})_\omega^* L^* \beta,$$

which is zero since β is primitive. Hence $(d_r)_\omega^* \beta$ is also primitive. This proves (3.11) and completes the proof of Proposition 3.6. \square

Now we will show several consequences of Proposition 3.6. Denote by Π_{pr} the projection $\Omega^*(M^{2n}) \rightarrow \mathcal{P}^*(M^{2n})$ according to the Lefschetz decomposition in (3.4). Set

$$d_k^+ := \Pi_{pr} d_k.$$

Using the first Lefschetz decomposition and Proposition 3.6 we decompose the operator $d_k : \Omega^q(M^{2n}) \rightarrow \Omega^{q+1}(M^{2n})$ for $0 \leq q \leq n$ as follows (cf. [29]).

$$(3.13) \quad d_k = d_k^+ + Ld_k^-,$$

where $d_k^- : \Omega^q(M^{2n}) \rightarrow \Omega^{q-1}(M^{2n})$, $0 \leq q \leq n$. Note that d_k^- is well-defined, since $L : \Omega^{q-1}(M^{2n}) \rightarrow \Omega^q(M^{2n})$ is injective. It is straightforward to check

$$(3.14) \quad d_k^+(\mathcal{L}^{s,r}) = 0 \text{ if } s \geq 1, \text{ and } d_k^-(\mathcal{L}^{s,r}) \subset \mathcal{L}^{s,r-1}.$$

Lemma 3.7. (cf. [29, Lemma 2.5, II]) *The operators d_k^+ , d_{k-1}^- satisfy the following properties*

$$(3.15) \quad (d_k^+)^2(\alpha^q) = 0,$$

$$(3.16) \quad d_{k-1}^- d_k^-(\alpha^q) = 0, \text{ if } q \leq n,$$

$$(3.17) \quad (d_k^- d_k^+ + d_{k-1}^+ d_k^-) \alpha^q = 0, \text{ if } q \leq n-1,$$

$$(3.18) \quad (d_{k-1})_\omega^* (d_k)_\omega^* (\alpha^q) = 0.$$

Proof. We use the equality $d_k^2 = 0$ in the form $d_k(d_k^+ + Ld_k^-) = 0$. Using (2.5) we get

$$(d_k^+)^2 + L(d_k^- d_k^+ + d_{k-1}^+ d_k^-) + L^2 d_{k-1}^- d_k^- = 0.$$

Now taking into account (3.14) and the injectivity of the operators $L : \Omega^q(M^{2n}) \rightarrow \Omega^{q+2}(M^{2n})$ and $L^2 : \Omega^{q-1}(M^{2n}) \rightarrow \Omega^{q+3}(M^{2n})$ for $q \leq n-1$ we obtain (3.15), (3.16), and (3.17).

Finally, (3.18) is a consequence of $d_k^2 = 0$ and $*_\omega^2 = Id$. \square

Proposition 3.6 and Lemma 3.7 lead to new cohomology groups and homology groups associated with a l.c.s. manifold (M^{2n}, ω, θ) . We observe that $\mathcal{P}^*(M^{2n})$ is stable under the action of the operators d_k^+ , $(d_k)_\omega^*$, d_k^- .

Definition 3.8. Assume that $0 \leq q \leq n-1$.

The *k-plus-primitive q-th cohomology group* of (M^{2n}, ω, θ) is defined by

$$(3.19) \quad H^q(\mathcal{P}^*(M^{2n}), d_k^+) := \frac{\ker d_k^+ : \mathcal{P}^q(M^{2n}) \rightarrow \mathcal{P}^{q+1}(M^{2n})}{d_k^+(\mathcal{P}^{q-1}(M^{2n}))}.$$

The *k-primitive q-th homology group* of (M^{2n}, ω, θ) is defined by

$$(3.20) \quad H_q(\mathcal{P}^*(M^{2n}), (d_k)_\omega^*) := \frac{\ker (d_k)_\omega^* : \mathcal{P}^q(M^{2n}) \rightarrow \mathcal{P}^{q-1}(M^{2n})}{(d_{k+1})_\omega^*(\mathcal{P}^{q+1}(M^{2n}))}.$$

The *k-minus-primitive q-th homology group* of (M^{2n}, ω, θ) is defined by

$$(3.21) \quad H_q(\mathcal{P}^*(M^{2n}), d_k^-) := \frac{\ker d_k^- : \mathcal{P}^q(M^{2n}) \rightarrow \mathcal{P}^{q-1}(M^{2n})}{d_{k+1}^-(\mathcal{P}^{q+1}(M^{2n}))}.$$

Now we show few simple properties of the associated (co)homology groups of a l.c.s. manifolds. Note that the formula (3.23) below has been proved in [29, Proposition 3.15] for compact symplectic manifolds (M^{2n}, ω) .

Proposition 3.9. *Assume that (M^{2n}, ω, θ) is a l.c.s. manifold, $n \geq 2$.*

1. *Suppose that $[(k-1)\theta] \neq 0 \in H^1(M^{2n}, \mathbb{R})$. Then*

$$(3.22) \quad H^1(\mathcal{P}^*(M^{2n}), d_k^+) = H^1(\Omega^*(M^{2n}), d_k).$$

2. *Suppose that $[(k-1)\theta] = 0 \in H^1(M^{2n}, \mathbb{R})$. Then*

$$(3.23) \quad H^1(\mathcal{P}^*(M^{2n}), d_k^+) = H^1(\Omega^*(M^{2n}), d_\theta) \text{ if } [\omega] \neq 0 \in H^2(\Omega^*(M^{2n}), d_\theta),$$

$$(3.24) \quad H^1(\mathcal{P}^*(M^{2n}), d_k^+) = H^1(\Omega^*(M^{2n}), d_\theta) \oplus R \text{ if } [\omega] = 0 \in H^2(\Omega^*(M^{2n}), d_\theta),$$

where R is the 1-dimensional vector space generated by $\rho \in H^1(\mathcal{P}^*(M^{2n}), d_k^+)$ with $d_k \rho = \omega$.

Proof. 1. Assume that $0 \neq \alpha \in \mathcal{P}^1(M^{2n})$ and $d_k^+ \alpha = 0$, i.e. $[\alpha] \in H^1(\mathcal{P}^*(M^{2n}), d_k^+)$. Since $d_k^+ \alpha = 0$ we get $d_k \alpha = Lf$, where $f \in C^\infty(M^{2n})$. Assume that $f \neq 0$. Using $d_k^2 \alpha = 0$ we get $Ld_{k-1}f = 0$, which implies $d_{k-1}f = 0$, since L is injective. The equality $d_{k-1}f = 0$ implies that d_{k-1} is gauge equivalent to d . This contradicts the assumption of Proposition 3.9.1. Hence $f = 0$. It follows $d_k \alpha = 0$. Using $d_k^+ h = d_k h$ for all $h \in \mathcal{P}^0(M^{2n})$ we get (3.22) immediately.

2. Now we assume that $[(k-1)\theta] = 0 \in H^1(M^{2n}, \mathbb{R})$, ore equivalently d_{k-1} is gauge equivalent to the canonical connection : $d_{k-1} = e^h d e^{-h} = d - dh \wedge$ for some $h \in C^\infty(M^{2n})$. In this case, as above, $d_k \alpha = Lf$ implies that $d_{k-1}f = 0$, and hence $f = ce^h$.

a) Assume that (3.23) holds. If $c \neq 0$, then $[e^h \omega] = 0 \in H^2(M^{2n}, d_k)$. Since $(k-1)\theta = -dh$ we get $d_\theta - dh \wedge$ is gauge equivalent to d_k . It follows that $[\omega] = 0 \in H^2(M^{2n}, d_\theta)$. This contradicts the assumption of (3.23). Hence $c = 0$. In this case we have $d_k \alpha = 0$, and therefore $[e^{-h} \alpha] \in H^1(M^{2n}, d_\theta)$. Taking into account $d_k^+ f = d_k f$ for any $f \in \mathcal{P}^0(M^{2n}) = \Omega^0(M^{2n})$ we obtain (3.23).

b) Now assume that (3.24) holds. Then $e^h \omega = d_k \rho$ for some $\rho \in \Omega^1(M^{2n})$. In this case $d_k(\alpha) = Lf = \omega c e^h$ implies that $d_k(\alpha - c\rho) = 0$. We conclude that if $d_k^+(\alpha) = 0$ then $\alpha = c\rho + \beta$ where $d_k(\beta) = 0$. Clearly $[\beta] \in H^1(M^{2n}, d_k) = H^1(M^{2n}, d_\theta)$. This proves (3.24) and completes the proof of Proposition 3.9. \square

Proposition 3.10. (cf. [29, Lemma 2.7, part II]) *Assume that $0 \leq k \leq n$. If $\alpha \in \mathcal{P}^k(M^{2n})$, then*

$$(3.25) \quad d_r^-(\alpha^k) = \frac{(d_r)_\omega^*(\alpha^k)}{n-k+1}.$$

Consequently $H_k(\mathcal{P}^*(M^{2n}), d_r^-) = H_k(\mathcal{P}^*(M^{2n}), (d_r)_\omega^*)$.

Proof. It suffices to prove (3.25) locally. Note that locally $d_\theta = d - df \wedge$. In this case $(d - df \wedge)\omega = 0$ implies $\omega = e^f \omega_0$ with $d\omega_0 = 0$. Next we compare $*_\omega$ and $*_{\omega_0}$ using (2.8) and the equality $G_\omega = e^{-f} G_{\omega_0}$.

$$\beta^k \wedge *_\omega \alpha^k = \wedge^k G_\omega(\beta^k, \alpha^k) \wedge \frac{\omega^n}{n!} = \wedge^k e^{-kf} G_{\omega_0}(\beta^k, \alpha^k) e^{nf} \frac{\omega_0^n}{n!} = e^{(n-k)f} \beta^k \wedge *_{\omega_0} \alpha^k,$$

where $\beta^k, \alpha^k \in \Omega^k(M^{2n})$. It follows that

$$(3.26) \quad *_\omega(\alpha^k) = e^{(n-k)f} *_{\omega_0}(\alpha^k).$$

Let $\alpha^k \in \mathcal{P}^k(M^{2n})$, $0 \leq k \leq n$. Denote by $(d)_{\omega_0}^*$ the symplectic adjoint of d with respect to ω_0 . The formula (3.27) below, which is a partial case of (3.25) for symplectic manifold, has been proved in [29, Lemma 2.7, part II]. (We observe that their operator d^Λ differs from our operator $(d)_{\omega_0}^*$ by sign (-1) .)

$$(3.27) \quad d^-(\alpha^k) = \frac{(d)_{\omega_0}^* \alpha^k}{n - k + 1}.$$

Using $d_{n+r-k}\alpha = e^{(n+r-k)f} d(e^{-(n+r-k)f} \alpha)$ we obtain from (3.26)

$$\begin{aligned} (d_r)_\omega^*(\alpha^k) &= (-1)^k *_\omega d_{n+r-k} *_\omega(\alpha^k) = \\ &= (-1)^k e^{(n-(2n-k+1))f} *_{\omega_0} e^{(n+r-k)f} d(e^{-(n+r-k)f} (e^{(n-k)f} *_{\omega_0} \alpha^k)) = \\ (3.28) \quad &= (-1)^k e^{(r-1)f} *_{\omega_0} d(e^{-rf} *_{\omega_0} \alpha^k) = \end{aligned}$$

$$(3.29) \quad = e^{-f} ((d)_{\omega_0}^* \alpha^k + (-1)^k *_{\omega_0} (-r) df \wedge *_{\omega_0} \alpha^k).$$

Substituting $r = 0$, we get from (3.29)

$$(3.30) \quad (d)_{\omega_0}^*(\alpha^k) = e^f (d)_\omega^* \alpha^k.$$

Next we compare d_r^- with d^- .

$$(3.31) \quad d_r(\alpha^k) = e^{rf} d(e^{-rf} \alpha^k) = e^{rf} [d^+(e^{-rf} \alpha^k) + \omega_0 \wedge d^- e^{-rf} \alpha^k].$$

Since the Lefschetz decomposition of $\Omega(M^{2n})$ is invariant under conformal transformations we get from (3.31)

$$(3.32) \quad d_r^-(\alpha^k) = e^{(r-1)f} d^-(e^{-rf} \alpha^k).$$

Combining (3.32) with (3.27) we get

$$(3.33) \quad d_r^-(\alpha^k) = e^{(r-1)f} \frac{(d)_{\omega_0}^* (e^{-rf} \alpha^k)}{n - k + 1}.$$

Taking into account (3.30) and (3.28) we obtain from (3.33)

$$(3.34) \quad d_r^-(\alpha^k) = e^{rf} \frac{(d)_\omega^* e^{-rf} \alpha^k}{n - k + 1} = \frac{(d_r)_\omega^* \alpha^k}{n - k + 1}.$$

This proves (3.25).

Clearly the second assertion of Proposition 3.10 follows from (3.25). This completes the proof of Proposition 3.10. \square

Let J be a compatible almost complex structure on a l.c.s. manifold (M^{2n}, ω, θ) . The complexified space $T_{\mathbb{C}}^*(M^{2n}) := (T^*(M^{2n}) \otimes \mathbb{C})$ is decomposed into eigen-subspaces $T^{p,q}(M^{2n})$. Let $\Pi^{p,q} : T_{\mathbb{C}}^*(M^{2n}) \rightarrow T^{p,q}(M^{2n})$ be the projection. Set

$$\mathcal{J} := \sum_{p,q} (\sqrt{-1})^{p-q} \Pi^{p,q}.$$

In what follows we want to apply the Hodge theory for compact l.c.s. manifold (M^{2n}, ω, θ) provided with a compatible metric g . First we derive a formula for the formal adjoint $(d_l^+)^*$ of $d_l^+ : \mathcal{P}^*(M^{2n}) := \mathcal{P}^*(M^{2n}) \rightarrow \mathcal{P}^*(M^{2n}) \subset \Omega^*(M^{2n})$. For any operator D acting on a subbundle $E \subset \Omega^*(M^{2n})$ we denote by $(D)^*$ the formal adjoint of D .

Lemma 3.11. *For any $\alpha \in \mathcal{P}^*(M^{2n})$ we have*

$$(3.35) \quad (d_l^+)^*(\alpha) = - *_g (d_{-l}) *_g (\alpha).$$

Proof. First, we want to compute the formal adjoint $(d_l)^*$ of $d_l = d + l\theta \wedge : \Omega^*(M^{2n}) \rightarrow \Omega^*(M^{2n})$. It is known that [32, §5.1.2]

$$(3.36) \quad (d)^* = - *_g d *_g.$$

Since $\theta \wedge$ is the symbol of d we get from (3.36)

$$(3.37) \quad (l\theta \wedge)^* = *_g l\theta \wedge *_g.$$

It follows from (3.36) and (3.37)

$$(3.38) \quad (d_l)^* = - *_g d_{-l} *_g.$$

Using (2.14) we get for $\alpha \in \mathcal{P}^*(M^{2n})$

$$(3.39) \quad (Ld_l^-)^*(\alpha) = (d_l^-)^* \Lambda(\alpha) = 0.$$

It follows from (3.38) and (3.39) that for $\alpha \in \mathcal{P}^*(M^{2n})$

$$(3.40) \quad (d_l^+)^*(\alpha) = - *_g (d_{-l}) *_g (\alpha).$$

This proves (3.35), which is consistent with [29, (3.2), part II], if (M^{2n}, ω) is a symplectic manifold. \square

Lemma 3.12. *(cf. [29, Lemma 3.4, part II]) Let J be a compatible almost complex structure on a l.c.s. manifold (M^{2n}, ω, θ) , g the associated compatible metric and $*_g$ the Hodge star operator with respect to g . Then for $\alpha^k, \alpha^{k-1} \in \mathcal{P}^*(M^{2n})$, $0 \leq k \leq n$, we have*

$$(3.41) \quad \mathcal{J}(d_l^+)^* \mathcal{J}^{-1}(\alpha^k) = (n - k + 1) d_{-l+k-n}^-(\alpha^k),$$

$$(3.42) \quad \mathcal{J} d_l^+ \mathcal{J}^{-1}(\alpha^{k-1}) = (n - k + 1) (d_{-l+k-n}^-)^*(\alpha^{k-1}).$$

Proof. Using [5, Theorem 2.4], see also Lemma 2.3, we get easily

$$(3.43) \quad \mathcal{J} = *_g *_\omega.$$

By (2.13) we get from (3.43)

$$(3.44) \quad \mathcal{J}^{-1}(\alpha^k) = *_\omega *_g (-1)^k (\alpha^k).$$

Combining (3.44) with (3.43), (3.35) and applying (2.11), (2.13) again we get

$$\begin{aligned} \mathcal{J}(d_l^+)_g \mathcal{J}^{-1}(\alpha^k) &= (-1)^{k+1} *_g *_\omega *_g d_{-l} *_g *_\omega *_g (\alpha^k) = \\ (3.45) \quad &= (-1)^{k+1} *_g^2 (d_{-l+k-n})_\omega^* (\alpha^k) = (d_{-l+k-n})_\omega^* (\alpha^k), \end{aligned}$$

since $*_\omega *_g = *_g *_\omega$. Using (3.25) we get (3.42) immediately from (3.45). Clearly (3.42) follows from (3.41), since they are adjoint. This completes the proof of Lemma 3.12. \square

The following Proposition is a generalization of [29, Proposition 3.5, part II].

Proposition 3.13. *Let (M^{2n}, ω, θ) be a compact l.c.s manifold. Then $H^k(\mathcal{P}^*(M^{2n}), d_l^+) = H_k(\mathcal{P}^*(M^{2n}), (d_{-l+k-n})_\omega^*)$ for all $l \in \mathbb{Z}$ and $0 \leq k \leq n-1$.*

Proof. First we note that all the operators d_l^+ , d_l^- and $(d_l)_\omega^*$ restricted to the space $\mathcal{P}^*(M^{2n})$ are elliptic operators. This observation is a consequence of the theorem by Bouche who proved that the complex of coeffective forms on a symplectic manifold M^{2n} is elliptic in dimension greater than n [4]. Indeed, the complex $(\mathcal{P}^*(M^{2n}), (d)_\omega^*)$ is dual to the complex of coeffective forms, see also Remark 3.15.1 below. Thus $(d_l)_\omega^*$ acting on $\mathcal{P}^*(M^{2n})$ is an elliptic operator, since $(d_l)_\omega^*$ has the same symbol as $(d)_\omega^*$. Taking (3.25), (3.42) and (3.41) into account we prove the ellipticity of d_l^- and d_l^+ acting on $\mathcal{P}^*(M^{2n})$. In [29, Proposition 2.8 part II] the authors give another proof of the ellipticity of these operators.

Now Proposition 3.13 follows easily from Lemma 3.12 using the Hodge theory. \square

Corollary 3.14. *Assume that (M^{2n}, ω, θ) is a connected compact twisted manifold. Then $H^0(\mathcal{P}^*(M^{2n}), d_k^+) = 0$ if d_k is not gauge equivalent to the canonical differential $d = d_0$, ore equivalently $[k\theta] \neq 0 \in H^1(M^{2n}, \mathbb{R})$. If otherwise, then $H^0(\mathcal{P}^*(M^{2n}), d_k^+) = H_0(\mathcal{P}^*(M^{2n}), (d_k)_\omega^*) = H_0(\mathcal{P}^*(M^{2n}), d_k^-) = \mathbb{R}$.*

Proof. Note that $\mathcal{P}^0(M^{2n}) = \Omega^0(M^{2n})$ and $d_k^+ = d_k$, which implies the first assertion of Proposition 3.14 immediately. The second assertion of Proposition 3.14 follows from Proposition 3.10 and Proposition 3.13, taking into account the equalities $H^0(\mathcal{P}^*(M^{2n}), d) = H^0(M^{2n}, \mathbb{R}) = \mathbb{R}$. \square

Remark 3.15. 1. Let (M^{2n}, ω, θ) be a l.c.s. manifold. The symplectic star operator $*_\omega$ provides an isomorphism between the space \mathcal{P}^*M^{2n} of primitive forms and the space \mathcal{C}^*M^{2n} of coeffective forms. Thus $H_*(\mathcal{P}^*(M^{2n}))$ is isomorphic to $H(\mathcal{C}^*M^{2n}, d_\theta)$. The latter cohomology groups for symplectic manifolds have been introduced by Bouche [4]. A variant of the effective cohomology groups for contact manifolds has been introduced (and computed) by Lychagin [21] already in 1979. Later, a modification of this complex for contact manifolds has been considered by Rumin independently

[25]. Chinea, Marrero and Leo generalized the construction of effective cohomology groups for Jacobi manifolds [6].

2. Note that the groups $H^q(\mathcal{P}^*(M^{2n}), d_k^+)$ have the following simple interpretation. We consider the differential ideal $L(\Omega^*(M^{2n})) \subset \Omega^*(M^{2n})$. The quotient $\Omega^*(M^{2n})/L(\Omega^*(M^{2n}))$ is isomorphic to the space $\mathcal{P}^*(M^{2n})$, and the differential d_k induces the differential d_k^+ on the quotient complex.

3. The plus-primitive cohomology groups and the minus-primitive cohomology groups for symplectic manifolds have been introduced by Tseng and Yau [29, part II].

4. Below we shall show a deeper relation between these new (co)homology groups and the twisted cohomology groups $H^*(M^{2n}, d_k)$ using the spectral sequence introduced in the next section.

5. Let (M^{2n+1}, α) be a contact manifold. Then its symplectization $M^{2n+2} := M^{2n+1} \times \mathbb{R}$ is supplied with a symplectic form $\omega(x, t) = \exp^t(d\alpha + dt \wedge \alpha) = \tilde{\alpha}$. Denote by $i : M^{2n+1} \rightarrow M^{2n+2}$ the embedding $x \mapsto (x, 0)$. We observe that the restriction of the filtration on $(M^{2n+2}, d\tilde{\alpha})$ to $i(M^{2n+1})$ coincides with the filtration introduced by Lychagin [21].

6. Note that any symplectic manifold (M^{2n}, ω) is a Poisson manifold equipped with the Poisson bivector G_ω . Associated with a Poisson bivector Λ on a Poisson manifold M there are two differential complexes. The first one is the complex of multivector fields on M equipped with the differential Λ acting via the Schouten bracket. It has been introduced by Lichnerowicz and the associated cohomology group is called the Lichnerowicz-Poisson cohomology of M [20], [10]. The second differential complex is the complex of differential forms on M equipped with the differential $\delta = [i(\Lambda), d]$ where $i(\Lambda)$ is the contraction with Λ . This complex has been introduced by Kozsul in [16] and it is called the canonical Poisson homology of M [5]. If M^{2n} is symplectic then $G_\omega \in \text{End}(T^*M^{2n}, TM^{2n})$ induces an isomorphism between the de Rham cohomology and Lichnerowicz-Poisson cohomology [10, Theorem 6.1], and the symplectic star operator provides an isomorphism between the de Rham cohomology and the canonical Poisson homology [5]. In [10] the authors consider the coeffective Lichnerowicz-Poisson cohomology groups on a Poisson manifold, which are isomorphic to the coeffective symplectic groups introduced by Bouche [4] if the Poisson structure is symplectic.

4. SPECTRAL SEQUENCES ON A L.C.S. MANIFOLD

In this section, we construct a spectral sequence associated with the filtered complex $(F^*K_k^*, d_k)$ on a l.c.s. manifold (M^{2m}, ω, θ) . We compare the E_1 -term of this spectral sequence with the primitive cohomological groups $H^{*,+}(\mathcal{P}^*(M^{2n}), d_k)$ introduced in the previous section (Lemma 4.1). We show the existence of a long exact sequence connecting the E_1 -term of this spectral sequence with the Lichnerowicz-Novikov cohomology groups $H^*(\Omega^*(M^{2n}), d_k)$, also denoted by $H_k^*(M^{2n})$ (Theorem 4.3, Proposition 4.5).

We prove that the E_1 -term of our spectral sequence is conformal invariant of (M^{2n}, ω, θ) , moreover the E_1 -terms associated with (M^{2n}, ω, θ) and $(M^{2n}, \omega', \theta)$ are isomorphic, if $\omega' = \omega + d_\theta \tau$ (Theorem 4.6).

In Proposition 3.5 of the previous section we proved that $(F^* K_k^*, d_k)$ is a filtered complex. Let us study the spectral sequence $\{E_{k,r}^{p,q}, d_{k,r} : E_{k,r}^{p,q} \rightarrow E_{k,r}^{p+r, q-r+1}\}$ of this filtered complex, first introduced by Di Pietro and Vinogradov for symplectic manifolds in [7]. We refer the reader to [23], [13] for an introduction into the theory of spectral sequences associated with a filtration. The initial term $E_{k,0}^{p,q}$ of this spectral sequence is defined as follows,

$$(4.1) \quad E_{k,0}^{p,q} = F^p K_k^{p+q} / F^{p+1} K_k^{p+q}.$$

Using the induced Lefschetz decomposition (3.6), (3.7), (3.8) and (3.9), taking into account the injectivity of the map $L^p : \Omega^{q-p}(M^{2n}) \rightarrow \Omega^{q+p}(M^{2n})$, we get for all $k \in \mathbb{Z}$

$$(4.2) \quad E_{k,0}^{p,q} \cong \mathcal{L}^{p, q-p} \cong \mathcal{L}^{0, q-p} \text{ if } n \geq q \geq p \geq 0,$$

$$(4.3) \quad E_{k,0}^{p,q} = 0 \text{ otherwise .}$$

Since $E_{k,l}^{p,q}$ is a quotient of $E_{k,l-1}^{p,q}$, in view of (4.3) any term $E_{k,l}^{p,q}$ written below, if without explicit condition on p and q , is always under the assumption $0 \leq p \leq q \leq n$ and $k \geq 0$.

Now let us go to the next term $E_{k,1}^{p,q}$ of our spectral sequence. Recall that $d_{k,0} : E_{k,0}^{p,q} \rightarrow E_{k,0}^{p,q+1}$ is obtained from the differential d_k by passing to the quotient:

$$(4.4) \quad \begin{array}{ccc} E_{k,0}^{p,q} & \xrightarrow{d_{k,0}} & E_{k,0}^{p,q+1} \\ \parallel & & \parallel \\ F^p K_k^{p+q} / F^{p+1} K_k^{p+q} & \longrightarrow & F^p K_k^{p+q+1} / F^{p+1} K_k^{p+q+1} \end{array}$$

Let us write $d_{k,0}$ explicitly using (4.2) and (4.3). Since $d_k L^p = L^p d_{k-p}$, using (4.1), (4.2), (4.3) and (3.13) we have for any $\alpha \in E_{k,0}^{p,q}$ with $n \geq q \geq p \geq 0$

$$(4.5) \quad d_{k,0}(\alpha) = [L^p(d_{k-p}^+ + Ld_{k-p}^-)(\tilde{\alpha})] = [L^p d_{k-p}^+(\tilde{\alpha})] \in E_{k,0}^{p,q+1},$$

where $\tilde{\alpha} \in \mathcal{L}^{0, q-p}$ is a representative of $\alpha \in E_{k,0}^{p,q}$ by (4.2). Since $L^p : \mathcal{L}^{0, q-p} \rightarrow \mathcal{L}^{p, q-p}$ is an isomorphism, if $0 \leq p \leq q \leq n$ by (4.2), we rewrite (4.5) as follows

$$(4.6) \quad d_{k,0} : E_{k,0}^{p,q} = \mathcal{L}^{0, q-p} \rightarrow E_{k,0}^{p,q+1} = \mathcal{L}_{k,0}^{0, q+1-p}, \quad \tilde{\alpha} \mapsto d_{k-p}^+ \tilde{\alpha},$$

if $0 \leq p \leq q \leq n$.

Lemma 4.1. *The term $E_{k,1}^{*,*}$ of the spectral sequence $\{E_{k,r}^{p,q}, d_{k,r}\}$ is determined by the following relations*

$$(4.7) \quad E_{k,1}^{p,q} = H^{q-p}(\mathcal{P}^*(M^{2n}), d_{k-p}^+) \text{ if } 0 \leq p \leq q \leq n-1,$$

$$(4.8) \quad E_{k,1}^{p,n} = \frac{\mathcal{P}^{n-p}(M^{2n})}{d_{k-p}^+(\mathcal{P}^{n-p-1}(M^{2n}))}, \text{ if } 0 \leq p \leq n,$$

$$(4.9) \quad E_{k,1}^{p,q} = 0 \text{ otherwise}.$$

Proof. Clearly (4.7) is a consequence of (4.6). Next using (4.5) and the identity $L^p(\mathcal{P}^{n-p+1}(M^{2n})) = 0$, we obtain $d_{k,0}(E_{k,0}^{p,n}) = 0$. Hence

$$(4.10) \quad E_{k,1}^{p,n} = \frac{L^p(\mathcal{P}^{n-p}(M^{2n}))}{L^p(d_{k-p}^+(\mathcal{P}^{n-p-1}(M^{2n})))}.$$

Since $L^p : \Omega^{n-p}(M^{2n}) \rightarrow \Omega^{n+p}(M^{2n})$ is injective, (4.10) implies (4.8).

The last relation (4.9) in Lemma 4.1 follows from (4.3). This completes the proof of Lemma 4.1. \square

Next we define the following diagram of chain complexes

$$\begin{array}{ccccccc} \Omega^{q-p-1}(M^{2n}) & \xrightarrow{L} & \Omega^{q-p+1}(M^{2n}) & \xrightarrow{\Pi L^p} & E_{l+p,0}^{p,q+1} & \longrightarrow & \\ \downarrow d_{l-1} & & \downarrow d_l & & \downarrow d_{l+p,0} & & \\ \Omega^{q-p}(M^{2n}) & \xrightarrow{L} & \Omega^{q-p+2}(M^{2n}) & \xrightarrow{\Pi L^p} & E_{l+p,0}^{p,q+2} & \longrightarrow & \\ \downarrow d_{l-1} & & \downarrow d_l & & \downarrow d_{l+p,0} & & \\ \Omega^{q-p+1}(M^{2n}) & \xrightarrow{L} & \Omega^{q-p+3}(M^{2n}) & \xrightarrow{\Pi L^p} & E_{l+p,0}^{p,q+3} & \longrightarrow & \end{array}$$

Here the map ΠL^p associates with each element $\beta \in \Omega^{q-p+1}(M^{2n})$ the element $[L^p \beta] \in E_{l+p,0}^{p,q+1}$. Recall that $d_l \circ L = L \circ d_{l-1}$. Thus the upper part of the above diagram is commutative. The lower part of the diagram is also commutative, since

$$d_{l+p} L^p = L^p d_l.$$

Summarizing we have the following sequence of chain complexes

$$(4.11) \quad 0 \rightarrow (\Omega^{q-(p+1)}(M^{2n}), d_{l-1}) \xrightarrow{L} (\Omega^{q+1-p}(M^{2n}), d_l) \xrightarrow{\Pi L^p} (E_{l+p,0}^{p,q+1}, d_{l+p,0}) \rightarrow 0.$$

Set $\Omega^{-1}(M^{2n}) := 0$.

Lemma 4.2. *The sequence (4.11) of chain complexes is exact for $0 \leq p \leq q \leq n-1$.*

Proof. For $0 \leq p \leq q \leq n$ the operator $L : \Omega^{q-p-1}(M^{2n}) \rightarrow \Omega^{q-p+1}(M^{2n})$ is injective by Lemma 3.4, so the sequence (4.11) is exact at $\Omega^{q-(p+1)}(M^{2n})$. The exactness at $\Omega^{q+1-p}(M^{2n})$ follows easily from Corollary 3.4, taking into

account (4.2). The exactness at $E_{l+p,0}^{p,q+1}$ follows directly from the definition (4.1) of $E_{l+p,0}^{p,q}$. \square

As a consequence of Lemma 4.2, using the general theory of homological algebra, see e.g. [12, Chapter 1, §6], we get immediately

Theorem 4.3. *The following long sequence is exact for $0 \leq p \leq q \leq n-1$*
(4.12)

$$\cdots \rightarrow H_l^{q-p}(M^{2n}) \xrightarrow{\bar{L}^p} E_{l+p,1}^{p,q} \xrightarrow{\delta_{p,q}} H_{l-1}^{q-(p+1)}(M^{2n}) \xrightarrow{\bar{L}} H_l^{q+1-p}(M^{2n}) \xrightarrow{\bar{L}^p} E_{l+p,1}^{p,q+1} \rightarrow \cdots$$

Remark 4.4. 1. Theorem 4.3 is a generalization of [7, Theorem 1] stated for symplectic manifolds.

2. Let us write the connecting homomorphism $\delta_{p,q}$ explicitly. If $\alpha \in H^{q-p}(\mathcal{P}^*(M^{2n}), d_l^+) = E_{l+p,1}^{p,q}$, see (4.7), then $d_l \tilde{\alpha} = L\eta$ for any representative $\tilde{\alpha} \in \mathcal{P}^{q-p}(M^{2n})$ of α . By the definition of connecting homomorphism, see also [7, §3], $\delta_{p,q}(\alpha) = [\eta] \in H_{l-1}^{q-(p+1)}(M^{2n})$. Since $d_l^+ \tilde{\alpha} = 0$, $d_l \tilde{\alpha} = Ld_l^- \tilde{\alpha}$, so $\eta = d_l^- \tilde{\alpha}$. Thus we get

$$(4.13) \quad \delta_{p,q}\alpha = [d_l^- \tilde{\alpha}] \in H_{l-1}^{q-(p+1)}(M^{2n}).$$

3. Let us write the operator \bar{L}^p explicitly. Assume that $[\beta] \in H_l^{q-p}(M^{2n})$, $\beta \in \Omega^{q-p}(M^{2n})$. Set

$$\beta_{pr} := \Pi_{pr}\beta$$

- the primitive component of β in the first Lefschetz decomposition. Since $d_l \beta = 0$, we have $d_l^+ \beta_{pr} = 0$. Thus the image $\bar{L}^p[\beta] = [L^p \beta] \in E_{l+p,1}^{p,q} = H^{q-p}(\mathcal{P}^*(M^{2n}), d_l^+)$ has a representative $\beta_{pr} \in \mathcal{P}^{q-p}(M^{2n})$ with $d_l^+ \beta_{pr} = 0$. Summarizing we have

$$(4.14) \quad \bar{L}^p[\beta] = [\beta_{pr}] \in H^{q-p}(\mathcal{P}^*(M^{2n}), d_l^+) = E_{l+p,1}^{p,q}.$$

4. Substituting for $0 \leq p = q \leq n-1$ in the long exact sequence (4.12), we get the left end of (4.12)

$$(4.15) \quad 0 \rightarrow H_l^0(M^{2n}) \rightarrow E_{l+p,1}^{p,p} \rightarrow 0 \rightarrow H_l^1(M^{2n}) \rightarrow \cdots$$

From (4.7) and (4.15) we get for $0 \leq p \leq n-1$

$$(4.16) \quad E_{l+p,1}^{p,p} = H^{0,+}(\mathcal{P}^*(M^{2n})) = H_l^0(M^{2n}).$$

Let us prolong the exact sequence (4.12) for $q = n$, using the ideas in [7, §III]. For $0 \leq k \leq n$ we set

$$C_l^k := \frac{\ker d_l^- \cap \Omega^k(M^{2n})}{d_l(\Omega^{k-1}(M^{2n}))}.$$

Proposition 4.5. *The long exact sequence (4.12) can be extended as follows*
(4.17)

$$E_{l+p,1}^{p,n-1} \xrightarrow{\delta_{p,n-1}} H_{l-1}^{n-(p+2)}(M^{2n}) \xrightarrow{[L]} C_l^{n-p} \xrightarrow{[L^p]} E_{l+p,1}^{p,n} \xrightarrow{\delta_{p,n}} C_{l-1}^{n-(p+1)} \xrightarrow{[L^{p+1}]} H_{l+p}^{n+p+1}(M^{2n}) \rightarrow 0,$$

where $0 \leq p \leq n-1$ and the operators $[L]$, $[L^p]$ and $[L^{p+1}]$ will be defined in the proof below.

Proof. First we define $[L]$ and prove the exactness at $H_{l-1}^{n-(p+2)}(M^{2n})$. Denote by $\Pi : H_l^{n-p}(M^{2n}) \rightarrow C_l^{n-p}$ the natural embedding of the quotient spaces

$$\frac{\ker d_l \cap \Omega^{n-p}(M^{2n})}{d_l(\Omega^{n-p-1}(M^{2n}))} \rightarrow \frac{\ker d_l^- \cap \Omega^{n-p}(M^{2n})}{d_l(\Omega^{n-p-1}(M^{2n}))}.$$

Set $[L] := \Pi \circ \bar{L}$, where $\bar{L} : H_{l-1}^{n-(p+2)}(M^{2n}) \rightarrow H_l^{n-p}(M^{2n})$ is induced by L . By Theorem 4.3 we have $\text{Im } \delta_{p,n-1} = \ker \bar{L}$. Since Π is an embedding, the last equality implies $\ker[L] = \ker \bar{L}$. This proves the required exactness at $H_{l-1}^{n-(p+2)}(M^{2n})$.

Now we define $[L^p]$ and show the exactness at C_l^{n-p} . Assume that $\alpha = \alpha_{pr} + L\tilde{\alpha} \in \Omega^{n-p}(M^{2n})$ is a representative of $[\alpha] \in C_l^{n-p}$, i.e. $d_l^-(\alpha) = 0$, or equivalently $d_l\alpha = d_l^+\alpha$. We set

$$(4.18) \quad [L^p](\alpha) := [\alpha_{pr}] \in \frac{\mathcal{P}^{n-p}(M^{2n})}{d_l^+(\mathcal{P}^{n-p-1}(M^{2n}))} = E_{l+p,1}^{p,n}.$$

Clearly the map $[L^p]$ is well-defined, since $\Pi_{pr}d_l(\gamma) = d_l^+\Pi_{pr}(\gamma)$. Now assume that $[\alpha] \in \ker[L^p]$, so by (7.6)

$$(4.19) \quad \Pi_{pr}\alpha = d_l^+\gamma \text{ for some } \gamma \in \mathcal{P}^{n-p-1}(M^{2n}).$$

Using the property $d_l\alpha = d_l^+\alpha$ we obtain from (4.19) $d_l\alpha = 0$. Now we write

$$(4.20) \quad \alpha = \alpha_{pr} + L\tilde{\alpha} = d_l^+\gamma + L\tilde{\alpha} = d_l\gamma + L\beta,$$

where $\beta = d_l^-\gamma + \tilde{\alpha}$. Since $d_l\alpha = 0$, using (4.20) we get $d_lL\beta = Ld_{l-1}\beta = 0$. Applying Lemma 3.4 to $d_{l-1}\beta \in \Omega^{n-p-1}(M^{2n})$, we obtain $d_{l-1}\beta = 0$. This implies $[\alpha] = [L]([\beta]) \in \text{Im } [L]$, and the required exactness.

Next we define the operator $\delta_{p,n} : E_{l+p,1}^{p,n} \rightarrow C_{l-1}^{n-(p+1)}$ as follows

$$(4.21) \quad \delta_{p,n}(\alpha) := [d_l^-\tilde{\alpha}] \in C_{l-1}^{n-(p+1)},$$

where $\tilde{\alpha} \in \mathcal{P}^{n-p}(M^{2n})$ is a representative of $\alpha \in E_{l+p,1}^{p,n} = \mathcal{P}^{n-p}(M^{2n})/d_l^+(\mathcal{P}^{n-p-1}(M^{2n}))$.

Clearly $\delta_{p,n}(\alpha) \in C_{l-1}^{n-p-1}$, since by (3.16) $d_{l-1}(d_l^-\tilde{\alpha}) = d_{l-1}^+d_l^-\tilde{\alpha}$. The map $\delta_{p,n}$ is well-defined, since for any $\gamma \in \mathcal{P}^{n-p-1}(M^{2n})$ using (3.16) and (3.17) we get

$$(4.22) \quad [d_l^-\tilde{\alpha}] = -[d_{l-1}^+d_l^-\tilde{\alpha}] = -[d_{l-1}(d_l^-\gamma)] = 0 \in C_{l-1}^{n-p-1}.$$

Now assume that $\alpha \in \ker \delta_{p,n}$ and $\tilde{\alpha} \in \mathcal{P}^{n-p}(M^{2n})$ is its representative. By (4.21) $d_l^-\tilde{\alpha} = d_{l-1}\beta$ for some $\beta \in \Omega^{n-p-2}(M^{2n})$. It follows

$$(4.23) \quad d_l^+\tilde{\alpha} = d_l\tilde{\alpha} - Ld_{l-1}\beta = d_l(\tilde{\alpha} - L\beta).$$

Since $\tilde{\alpha}$ is primitive, and $d_l^+(\mathcal{L}^{s,r}) \subset \mathcal{L}^{s,r+1}$, we get

$$(4.24) \quad d_l^+\tilde{\alpha} = d_l^+(\tilde{\alpha} - L\beta).$$

Clearly, (4.23) and (4.24) imply $[\tilde{\alpha} - L\beta] \in C_l^{n-p}$, and by (7.6) $\alpha = [L^p][\tilde{\alpha} - L\beta]$. This yields the exactness at $E_{l+p,1}^{p,n}$.

Let us define $[L^{p+1}]$ and show the exactness at $C_{l-1}^{n-(p+1)}$. For $\alpha \in C_{l-1}^{n-p-1}$ we set

$$(4.25) \quad [L^{p+1}](\alpha) := [L^{p+1}\tilde{\alpha}] \in H_{l+p}^{n+p+1}(M^{2n}),$$

where $\tilde{\alpha} \in \Omega^{n-p-1}(M^{2n})$ is a representative of α . Note that $d_{l+p}L^{p+1}\tilde{\alpha} = L^{p+1}d_{l-1}\tilde{\alpha} = 0$, so $[L^{p+1}](\alpha) \in H_{l+p}^{n+p+1}(M^{2n})$. The same formula shows that our map $[L^{p+1}]$ does not depend on the choice of a representative $\tilde{\alpha}$ of α . Now assume that $\alpha \in \ker[L^{p+1}]$. Then $L^{p+1}\tilde{\alpha} = d_{l+p}\beta$ for some $\beta \in \Omega^{n+p}(M^{2n})$. Using the Lefschetz decomposition for β we write

$$\beta = L^p(\beta_0 + \sum_{k=1}^{\lfloor \frac{n+p}{2} \rfloor} L^k \beta_k), \beta_i \in \mathcal{P}^*(M^{2n}).$$

It follows that

$$(4.26) \quad L^{p+1}\tilde{\alpha} = d_{l+p}\beta = L^p(d_l^+ \beta_0 + L(d_l^- \beta_0 + d_{l-1}(\beta_1 + L\beta_2 + \cdots))).$$

By Corollary 3.4.1 $L^{p+1} : \Omega^{n-p-1}(M^{2n}) \rightarrow \Omega^{n+p+1}(M^{2n})$ is an isomorphism, hence we get from (4.26)

$$(4.27) \quad \tilde{\alpha} = d_l^- \beta_p + d_{l-1}(\beta_{p+1} + L\beta_{p+1} + \cdots).$$

Combining (4.27) with (4.21) we get $\alpha \in \text{Im } \delta_{p,n}$. This proves the required exactness.

Finally we show that $[L^{p+1}]$ is surjective. Assume that $\beta \in \Omega^{n+p+1}(M^{2n})$ is a representative of $[\beta] \in H_{l+p+1}^{n+p+1}(M^{2n})$. Using the Lefschetz decomposition we write $\beta = L^{p+1}(\tilde{\beta})$, $\tilde{\beta} \in \Omega^{n-p-1}(M^{2n})$. Note that $L^{p+2}d_l^- \tilde{\beta} = Ld_{l+p}^- \beta = 0$, since $d_{l+p}\beta = 0$. Since $L^{p+2} : \Omega^{n-p-2}(M^{2n}) \rightarrow \Omega^{n+p+2}(M^{2n})$ is an isomorphism, we get $d_l^- \tilde{\beta} = 0$. Hence $[\tilde{\beta}] \in C_l^{n-p-1}$ and $[\beta] = [L^{p+1}][\tilde{\beta}]$. This completes the proof of Proposition 4.5. \square

Theorem 4.6. (cf. [8, Osservazione 18]) *The spectral sequences $E_{k,r}^{p,q}$ on (M^{2n}, ω, θ) and on $(M^{2n}, \omega', \theta')$ are isomorphic, if ω and ω' are conformal equivalent. Furthermore, the $E_{k,1}^{*,*}$ -terms of the spectral sequences on (M, ω, θ) and (M, ω', θ') are isomorphic, if $\omega' = \omega + d_\theta \rho$ for some $\rho \in \Omega^1(M^{2n})$.*

Proof. If $\omega' = \pm e^f \omega$, then $d_\theta \omega' = df \wedge \omega'$. Hence

$$(4.28) \quad (d_\theta - df \wedge) \omega' = 0.$$

Since $L : \Omega^1(M^{2n}) \rightarrow \Omega^3(M^{2n})$ is injective, (4.28) implies that

$$(4.29) \quad d_{\theta'} = d_\theta - df \wedge.$$

It follows

$$(4.30) \quad d_{k\theta'} = d_{k\theta} - k \cdot df \wedge.$$

Hence the map $I_f : \alpha \mapsto e^{kf} \alpha$ is an isomorphism between complexes $(F^* K_k^*, d_{k\theta})$ and $(F^* K_{k'}^*, d_{k'\theta'})$, since

$$(4.31) \quad d_{k\theta'}(e^{kf} \alpha) = d_{k\theta}(e^{kf} \alpha) + (-k \cdot df) \wedge e^{kf} \alpha = e^{kf}(d_{k\theta} \alpha).$$

It follows that the resulting terms $E_{k,0}^{p,q}$ are also conformal equivalent. Moreover, the map I_f induces an isomorphism I_f^0 between complexes

$$I_f^0 : (E_{k,0}^{p,q}, d_{k,0}) \rightarrow (E_{k',0}^{p,q}, d'_{k',0}), [\alpha] \mapsto [e^{kf} \alpha].$$

Inductively, this proves the first assertion of Theorem 4.6.

Now we assume that $\omega' = \omega + d_\theta \rho$. Then $d_\theta \omega' = 0$. Using the injectivity of $L : \Omega^1(M^{2n}) \rightarrow \Omega^3(M^{2n})$ we conclude that the Lee form of ω' is equal to the Lee form θ of ω . Denote by L' the wedge product with ω' , and by $[L']$ the induced operator on $H_l^*(M^{2n})$. Using $\omega' - \omega = d_\theta \rho$ and applying (2.6), which implies that the wedge product with $d_\theta \rho$ maps $H_l^k(M^{2n})$ to zero, we conclude that the operators L and L' induce the same map $H_l^k(M^{2n}) \rightarrow H_{l+1}^{k+2}(M^{2n})$.

To prove the second assertion of Theorem 4.6 we use the following version of Five Lemma, whose proof is obvious and hence omitted.

Lemma 4.7. *Assume that the following diagram of vector spaces A_i, B_i over a field \mathbb{F} is commutative. If the rows are exact and $A_1 \rightarrow B_1$, $A_2 \rightarrow B_2$, $A_4 \rightarrow B_4$, $A_5 \rightarrow B_5$ are isomorphisms, then there is an isomorphism from A_3 to B_3 , which also commutes with the other arrows.*

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \end{array}$$

The second assertion of Theorem 4.6 for $E_{k,1}^{p,q}$ follows immediately from the long exact sequence (4.12) and the formula (4.31), if $0 \leq p \leq q \leq n-1$, taking into account Lemma 4.1.

To examine the term $E_{k,1}^{p,n}$, $0 \leq p \leq n-1$, we need the following

Lemma 4.8. *Assume that $\omega' = \omega + d_\theta \rho$. For $0 \leq p \leq n$ there are linear maps $B_l^{n-p} : C_l^{n-p}(\omega) \rightarrow C_l^{n-p}(\omega')$ such that the following two diagrams are commutative. (The symbol I denotes the identity mapping. The other mapping are defined in the proof.)*

$$\begin{array}{ccc} H_{l-1}^{n-(p+2)}(M^{2n}) \xrightarrow{[L]} C_l^{n-p}(\omega) & & C_{l-1}^{n-(p+1)}(\omega) \xrightarrow{[L^{p+1}]} H_{l+p}^{n+p+1}(M^{2n}) \\ \downarrow I & & \downarrow B_{l-1}^{n-p-1} \\ H_{l-1}^{n-(p+2)}(M^{2n}) \xrightarrow{[L']} C_l^{n-p}(\omega') & & C_{l-1}^{n-(p+1)}(\omega') \xrightarrow{[L'^{p+1}]} H_{l+p}^{n+p+1}(M^{2n}) \end{array}$$

Proof. Let us define first a linear mapping

$$\tilde{B}_l^{n-p} : \ker d_l^-(\omega) \cap \Omega^{n-p}(M^{2n}) \rightarrow \Omega^{n-p}(M^{2n}).$$

Let $\eta \in \ker d_l^- \cap \Omega^{n-p}(M^{2n})$. This means that $d_l \eta = d_l^+ \eta$ or equivalently that $d_l \eta$ is primitive. The last assertion is again equivalent to the equality $\omega^p \wedge d_l \eta = 0$. Since $(L')^p : \Omega^{n-p}(M^{2n}) \rightarrow \Omega^{n+p}(M^{2n})$ is an isomorphism, there is a unique $\eta' \in \Omega^{n-p}(M^{2n})$ such that

$$\sum_{i=1}^p \binom{p}{i} \rho \wedge (d_\theta \rho)^{i-1} \wedge \omega^{p-i} \wedge d_l \eta = \omega'^p \wedge \eta'.$$

Now we define \tilde{B}_l^{n-p} by

$$\tilde{B}_l^{n-p} \eta := \eta - \eta'.$$

We shall now prove that the element $d_l(\eta - \eta')$ is ω' -primitive.

$$\begin{aligned} \omega'^p \wedge d_l(\eta - \eta') &= -\omega'^p \wedge d_l \eta' + (\omega + d_1 \rho)^p \wedge d_l \eta = \\ &= -d_{p+l}(\omega'^p \wedge \eta') + \omega^p \wedge d_l \eta + \sum_{i=1}^p \binom{p}{i} (d_1 \rho)^i \wedge \omega^{p-i} \wedge d_l \eta = \\ &= -d_{p+l}(\omega'^p \wedge \eta') + d_{p+l} \left[\sum_{i=1}^p \binom{p}{i} \rho \wedge (d_1 \rho)^{i-1} \wedge \omega^{p-i} \wedge d_l \eta \right] = \\ &= d_{p+l} \left[-\omega'^p \wedge \eta' + \sum_{i=1}^p \binom{p}{i} \rho \wedge (d_1 \rho)^{i-1} \wedge \omega^{p-i} \wedge d_l \eta \right] = 0. \end{aligned}$$

In other words we have proved that \tilde{B}_l^{n-p} maps $\ker d_l^- \cap \Omega^{n-p}(M^{2n})$ into $\ker (d')_l^- \cap \Omega^{n-p}(M^{2n})$, where $(d')_l^-$ is defined via the Lefschetz decomposition corresponding to ω' .

Let us take now an element $\eta \in d_l \Omega^{n-p-1}(M^{2n})$, i. e. $\eta = d_l \gamma$, where $\gamma \in \Omega^{n-p-1}(M^{2n})$. Then $d_l \eta = 0$ and we have $\omega'^p \wedge \eta' = 0$, which implies $\eta' = 0$. We thus get $\tilde{B}_l^{n-p} \eta = \eta$. Consequently we have proved that \tilde{B}_l^{n-p} maps $d_l(\Omega^{n-p-1}(M^{2n}))$ into itself. Now it is obvious that \tilde{B}_l^{n-p} induces a linear mapping

$$B_l^{n-p} : C_l^{n-p}(\omega) \rightarrow C_l^{n-p}(\omega').$$

Next we shall investigate the first diagram. First we define the mapping $[L]$. If $[\beta] \in H_{l-1}^{n-(p+2)}(M^{2n})$, then we have an element $\beta \in \Omega^{n-(p+2)}(M^{2n})$ such that $d_{l-1} \beta = 0$. Let us set

$$[L][\beta] := [\omega \wedge \beta].$$

It is easy to see that this definition depends only on the cohomology class $[\beta] \in H_{l-1}^{n-p-2}(M^{2n})$. Namely, if $\tilde{\beta} = \beta + d_{l-1} \gamma$, then

$$\omega \wedge (\beta + d_{l-1} \gamma) = \omega \wedge \beta + d_l(\omega \wedge \gamma),$$

which shows that $[L][\tilde{\beta}] = [L][\beta]$. Similarly we define $[L']$. Let us take $[\beta] \in H_{l-1}^{n-(p+2)}(M^{2n})$. Then $d_{l-1}\beta = 0$, and we have

$$\begin{aligned} & \sum_{i=1}^p \binom{p}{i} \rho \wedge (d_1 \rho)^{i-1} \wedge \omega^{p-i} \wedge d_l(\omega \wedge \beta) = \\ &= \sum_{i=1}^p \binom{p}{i} \rho \wedge (d_1 \rho)^{i-1} \wedge \omega^{p-i} \wedge (d_1 \omega \wedge \beta + \omega \wedge d_{l-1} \beta) = 0. \end{aligned}$$

We have $0 = \omega'^p \wedge \eta'$, which implies $\eta' = 0$. Thus we get $B_l^{n-p}[L][\beta] = B_l^{n-p}[\omega \wedge \beta] = [\omega \wedge \beta - 0] = [\omega \wedge \beta]$. On the other hand we compute

$$\begin{aligned} [L'][\beta] &= [\omega' \wedge \beta] = [(\omega + d_1 \rho) \wedge \beta] = [\omega \wedge \beta] + [d_1 \rho \wedge \beta + \rho \wedge d_{l-1} \beta] = \\ &= [\omega \wedge \beta] + [d_l(\rho \wedge \beta)] = [\omega \wedge \beta]. \end{aligned}$$

We have thus shown that the first diagram is commutative.

We continue now with the second diagram. Again we define first the mapping $[L^{p+1}]$. For $[\eta] \in C_{l-1}^{n-(p+1)}(\omega)$ there is a representative $\eta \in \Omega_{l-1}^{n-(p+1)}(M^{2n})$ such that $d_{l-1}^- \eta = 0$. This means that $d_{l-1} = d_{l-1}^+ \eta$. We compute

$$\begin{aligned} d_{l+p}(\omega^{p+1} \wedge \eta) &= d_{(p+1)+(l-1)}(\omega^{p+1} \wedge \eta) = d_{p+1}(\omega^{p+1}) \wedge \eta + \omega^{p+1} \wedge d_{l-1} \eta = \\ &= \omega^{p+1} \wedge d_{l-1}^+ \eta = 0. \end{aligned}$$

The last term vanishes because the form $d_{l-1}^+ \eta$ is primitive. Finally let us suppose that $\eta = d_{l-1} \gamma$. Then we have

$$\omega^{p+1} \wedge d_{l-1} \gamma = d_{p+1}(\omega^{p+1}) \wedge \gamma + \omega^{p+1} \wedge d_{l-1} \gamma = d_{(p+1)+(l-1)}(\omega^{p+1} \wedge \gamma).$$

This shows that we can define $[L^{p+1}]$ by the formula

$$[L^{p+1}][\eta] := [\omega^{p+1} \wedge \eta].$$

Now we are going to prove the commutativity of the second diagram. For $[\eta] \in C_{l-1}^{n-(p+1)}(\omega)$ we have $B_{l-1}^{n-p-1}[\eta] = [\eta - \eta']$, where η' is uniquely determined by the equality

$$\omega'^{p+1} \wedge \eta' = \sum_{i=1}^{p+1} \binom{p+1}{i} \rho \wedge (d_1 \rho)^{i-1} \wedge \omega^{p+1-i} \wedge d_{l-1} \eta.$$

Further we have $[L']B_{l-1}^{n-p-1}[\eta] = [\omega' \wedge (\eta - \eta')]$. Now let us compute

$$\begin{aligned}
& \omega'^{p+1} \wedge (\eta - \eta') - \omega^{p+1} \wedge \eta = \omega'^{p+1} \wedge \eta - \omega'^{p+1} \wedge \eta' - \omega^{p+1} \wedge \eta = \\
& = (\omega + d_1 \rho)^{p+1} \wedge \eta - \sum_{i=1}^{p+1} \binom{p+1}{i} \rho \wedge (d_1 \rho)^{i-1} \wedge \omega^{p+1-i} \wedge d_{l-1} \eta - \omega^{p+1} \wedge \eta = \\
& = \sum_{i=1}^{p+1} \binom{p+1}{i} (d_1 \rho)^i \wedge \omega^{p+1-i} \wedge \eta - \sum_{i=1}^{p+1} \binom{p+1}{i} \rho \wedge (d_1 \rho)^{i-1} \wedge \omega^{p+1-i} \wedge d_{l-1} \eta = \\
& = d_{p+1} \left(\sum_{i=1}^{p+1} \binom{p+1}{i} \rho \wedge (d_1 \rho)^{i-1} \wedge \omega^{p+1-i} \right) \wedge \eta \\
& \quad - \left(\sum_{i=1}^{p+1} \binom{p+1}{i} \rho \wedge (d_1 \rho)^{i-1} \wedge \omega^{p+1-i} \right) \wedge d_{l-1} \eta = \\
(4.32) \quad & = d_{(p+1)+(l-1)} \left(\sum_{i=1}^{p+1} \binom{p+1}{i} \rho \wedge (d_1 \rho)^{i-1} \wedge \omega^{p+1-i} \wedge \eta \right).
\end{aligned}$$

We have thus proved that $[L'^{p+1}]B_{l-1}^{n-p-1}[\eta] = [L^{p+1}][\eta]$. \square

Clearly the second assertion of Proposition 4.6 for $E_{k,1}^{p,n}$, $0 \leq p \leq n-1$, follows from Proposition 4.5, Lemma 4.7, and Lemma 4.8. Combining with (4.8), which implies that $E_{k,1}^{n,n} = C^\infty(M^{2n})$, we obtain the second assertion of Proposition 4.6. This completes the proof of Theorem 4.6. \square

Remark 4.9. In [11, Example 7.1] the authors construct an example of a compact 6-dimensional nilmanifold M^6 equipped with a family of symplectic forms ω_t , $t \in [0, 1]$, with varying cohomology classes $[\omega_t] \in H^2(M^6, \mathbb{R})$. They showed that the coeffective cohomology groups associated to ω_1 and ω_2 have different Betti number b_4 . It follows that, using [29, Lemma 2.7, Proposition 3.5, part II], see also Remark 3.15.1 above, the E_1 -terms of the associated spectral sequences for ω_0 and ω_1 are different.

5. THE STABILIZATION OF THE SPECTRAL SEQUENCES

In this section we prove that the spectral sequences $\{E_{l,r}^{p,q}\}$ on l.c.s.manifolds (M^{2n}, ω, θ) converge to the Lichnerowicz-Novikov cohomology $H_l^*(M^{2n})$ at the second term $E_{l,2}^{*,*}$, or at the t -term $E_{l,t}^{*,*}$ under some cohomological conditions posed on ω (Theorems 5.2, 5.8, 5.13). As a consequence, we obtain a relation between the primitive cohomology groups and the de Rham cohomology groups of (M^{2n}, ω) , if (M^{2n}, ω) is a compact symplectic manifold. This gives an answer to the question posed by Tseng and Yau in [29], see Remark 5.18.

First we prove the following simple property of the second terms $E_{l,2}^{*,*}$, which will be used later in the proof of Theorem 5.13.

Proposition 5.1. (cf. [8, Proposizione 19]) *Assume that $1 \leq p \leq q \leq n-1$. Then $E_{l,2}^{p,q} = E_{l,2}^{p-1,q-1}$.*

Proof. Let $\alpha \in E_{l+p,1}^{p,q} = H^{q-p}(\mathcal{P}^*(M^{2n}), d_l^+)$ and $\tilde{\alpha} \in \mathcal{P}^{q-p}(M^{2n})$ its representative as in (4.7). The differential $d_{l+p,1} : E_{l+p,1}^{p,q} \rightarrow E_{l+p,1}^{p+1,q}$ is defined by

$$(5.1) \quad d_{l+p,1}(\alpha) := [d_{l+p} L^p \tilde{\alpha}] = [L^p d_l \tilde{\alpha}] \in E_{l+p,1}^{p+1,q}.$$

Using $d_{l+p} = d_{l+p}^+ + L d_{l+p}^-$ and taking into account $d_l^+ \tilde{\alpha} = 0$, we observe that $[L^p d_l \tilde{\alpha}] \in E_{l+p,1}^{p+1,q}$ has a representative $d_l^-(\tilde{\alpha}) \in \mathcal{P}^{q-p-1}(M^{2n})$ in $H^{q-p-1}(\mathcal{P}^*(M^{2n}), d_{l-1}^+) = E_{l+p,1}^{p+1,q}$, if $0 \leq p \leq q \leq n$. Equivalently, using (4.7), we rewrite $d_{l+p,1}$ for $0 \leq p \leq q \leq n$ as follows

$$(5.2) \quad d_{l+p,1} : H^{q-p}(\mathcal{P}^*(M^{2n}), d_l^+) \rightarrow H^{q-p-1}(\mathcal{P}^*(M^{2n}), d_{l-1}^+), [\tilde{\alpha}] \mapsto [d_l^- \tilde{\alpha}].$$

Clearly Proposition 5.1 follows from (4.7) and the formula (5.1), (5.2). \square

Now assume that $\omega = d_k \tau$ for some $k \in \mathbb{Z}$ and $\tau \in \Omega^1(M^{2n})$. Since $d_1 \omega = d_k \omega = 0$, it follows that $(k-1)\theta \wedge \omega = 0$. Since L is injective, we get $k = 1$. The following theorem is a generalization of [7, Theorem 2] for the symplectic case $\theta = 0$.

Theorem 5.2. (cf. [7, Theorem 2]) *Assume that $\omega = d_1 \tau$. Then $E_{l,2}^{p,q} = 0$, if $1 \leq p \leq q \leq n-1$. If $0 \leq q \leq n$, then $E_{l,2}^{0,q} = H_l^q(M^{2n})$. If $0 \leq p \leq n$ then $E_{l+p,2}^{p,n} = H_{l+p}^{n+p}(M^{2n})$. Thus the spectral sequence $\{E_{l,r}^{p,q}, d_{l,r}\}$ stabilizes at the term $E_{l,2}$.*

Proof. Assume that $\omega = d_1 \tau$. Then for any d_{l-1} -closed form α we have

$$(5.3) \quad d_1 \tau \wedge \alpha = d_l(\tau \wedge \alpha).$$

Hence the induced operator \bar{L} in the exact sequence (4.12) satisfies

$$(5.4) \quad \bar{L}(H_{l-1}^{q-(p+1)}(M^{2n})) = 0 \in H_l^{q+1-p}(M^{2n}).$$

The equality (5.4) and the exact sequence (4.12) lead to the following exact sequence for $0 \leq p \leq q \leq n-1$.

$$(5.5) \quad 0 \rightarrow H_l^{q-p}(M^{2n}) \xrightarrow{\bar{L}^p} E_{l+p,1}^{p,q} \xrightarrow{\delta_{p,q}} H_{l-1}^{q-(p+1)}(M^{2n}) \rightarrow 0.$$

Using the isomorphism $E_{l+p,1}^{p,q} = H^{q-p}(\mathcal{P}^*(M^{2n}), d_l^+)$ and the formulae (4.13) and (4.14) describing $\delta_{p,q}$ and \bar{L}^p we rewrite the exact sequence (5.5) as follows

$$(5.6) \quad 0 \rightarrow H_l^{q-p}(M^{2n}) \xrightarrow{[\Pi_{pr}]} H^{q-p}(\mathcal{P}^*(M^{2n}), d_l^+) \xrightarrow{[d_l^-]} H_{l-1}^{q-(p+1)}(M^{2n}) \rightarrow 0.$$

The proof of the first and second assertion of Theorem 5.2 is based on our analysis of the long exact sequence of cohomology groups associated with the

short exact sequence (5.6). By (5.2), for $0 \leq p \leq q \leq n-1$, the differential $d_{l+p,1} : E_{l+p,1}^{p,q} \rightarrow E_{l+p,1}^{p+1,q}$ induces the following boundary operator

$$(5.7) \quad \hat{d}_l^- : H^{q-p}(\mathcal{P}^*(M^{2n}), d_l^+) \rightarrow H^{q-p-1}(\mathcal{P}^*(M^{2n}), d_{l-1}^+), [\tilde{\alpha}] \mapsto [d_l^- \tilde{\alpha}],$$

for $\tilde{\alpha} \in \mathcal{P}^{q-p}(M^{2n})$.

Lemma 5.3. *The short exact sequence (5.6) generates a short exact sequence of the following chain complexes for $1 \leq p \leq q \leq n-1$*

$$(5.8) \quad 0 \rightarrow (H_l^{q-p}(M^{2n}), \tilde{d}_l := 0) \xrightarrow{[\Pi_{pr}]} (H^{q-p}(\mathcal{P}^*(M^{2n}), d_l^+), \hat{d}_l^-) \rightarrow \xrightarrow{[d_l^-]} (H_{l-1}^{q-(p+1)}(M^{2n}), \tilde{d}_{l-1} := 0) \rightarrow 0.$$

Proof. It is useful to rewrite the sequence (5.8) of chain complexes as the following diagram

$$\begin{array}{ccccccc} H_{l+1}^{q-(p-1)}(M^{2n}) & \xrightarrow{[\Pi_{pr}]} & H^{q-(p-1)}(\mathcal{P}^*(M^{2n}), d_{l+1}^+) & \xrightarrow{[d_{l+1}^-]} & H_l^{q-p}(M^{2n}) & \longrightarrow & \\ \downarrow \tilde{d}_{l+1}=0 & & \downarrow \hat{d}_{l+1}^- & & \downarrow \tilde{d}_l=0 & & \\ H_l^{q-p}(M^{2n}) & \xrightarrow{[\Pi_{pr}]} & H^{q-p}(\mathcal{P}^*(M^{2n}), d_l^+) & \xrightarrow{[d_l^-]} & H_{l-1}^{q-(p+1)}(M^{2n}) & \longrightarrow & \\ \downarrow \tilde{d}_l=0 & & \downarrow \hat{d}_l^- & & \downarrow \tilde{d}_{l-1}=0 & & \\ H_{l-1}^{q-(p+1)}(M^{2n}) & \xrightarrow{[\Pi_{pr}]} & H^{q-(p+1)}(\mathcal{P}^*(M^{2n}), d_{l-1}^+) & \xrightarrow{[d_{l-1}^-]} & H_{l-2}^{q-(p+2)}(M^{2n}) & \longrightarrow & \end{array}$$

To prove Lemma 5.3 it suffices to show that the above diagram is commutative, or equivalently

$$(5.9) \quad \hat{d}_l^- [\Pi_{pr}] = \tilde{d}_l [\Pi_{pr}] = 0,$$

$$(5.10) \quad [d_{l-1}^-] \hat{d}_l^- = \tilde{d}_{l-1} [d_l^-] = 0.$$

Let $\alpha \in H_l^{q-p}(M^{2n})$ and $\tilde{\alpha} \in \Omega^{q-p}(M^{2n})$ its representative. Let $\tilde{\alpha} = \tilde{\alpha}_{pr} + L\tilde{\beta}_{pr} + L^2\gamma$ be the Lefschetz decomposition of $\tilde{\alpha}$. Using $d_l\tilde{\alpha} = d_l^+\tilde{\alpha} = d_l^+\tilde{\alpha}_{pr} = 0$ we obtain $Ld_l^-\tilde{\alpha}_{pr} = Ld_{l-1}^+\tilde{\beta}_{pr} + L^2(d_{l-1}^-\tilde{\beta}_{pr} + d_{l-2}\gamma)$. Hence

$$0 = L(d_l^-\tilde{\alpha}_{pr} + d_{l-1}^+\tilde{\beta}_{pr}) + L^2(d_{l-1}^-\tilde{\beta}_{pr} + d_{l-2}\gamma),$$

which implies $d_l^-\tilde{\alpha}_{pr} + d_{l-1}^+\tilde{\beta}_{pr} = 0$ thanks to the uniqueness of the second Lefschetz decomposition. It follows

$$(5.11) \quad \hat{d}_l^- [\Pi_{pr}]\alpha = \hat{d}_l^- [\tilde{\alpha}_{pr}] = [d_l^-\tilde{\alpha}_{pr}] = -[d_{l-1}^+\tilde{\beta}_{pr}] = 0 \in H^{q-(p+1)}(\mathcal{P}^*(M^{2n}), d_{l-1}^+).$$

Let $\beta \in H^{q-p}(\mathcal{P}^*(M^{2n}), d_l^+)$ and $\tilde{\beta} \in \mathcal{P}^{q-p}(M^{2n})$ its representative. Then

$$(5.12) \quad [d_{l-1}^-] \hat{d}_l^- \beta = [d_{l-1}^- d_l^+ \tilde{\beta}] = 0 \in H_{l-2}^{q-(p+2)}(M^{2n}).$$

Clearly (5.21) and (5.22) follow from (5.23) and (5.12). This completes the proof of Lemma 5.3. \square

The short exact sequence (5.8) in Lemma 5.3 generates the following associated long exact sequence of the cohomology groups

(5.13)

$$\rightarrow E_{l+p,2}^{p-1,q} \rightarrow H_l^{q-p}(M^{2n}) \xrightarrow{\delta} H_l^{q-p}(M^{2n}) \rightarrow E_{l+p,2}^{p,q} \rightarrow H_{l-1}^{q-(p+1)}(M^{2n}) \xrightarrow{\delta}$$

where δ is the connecting homomorphism.

Lemma 5.4. *We have $\delta(x) = x$ for all $x \in H_{l-1}^{q-(p+1)}(M^{2n})$ and for all $1 \leq p \leq n-1$.*

Proof of Lemma 5.6. Let $x \in H_{l-1}^{q-(p+1)}(M^{2n})$. Using (5.8) we write $x = [d_l^-]y$, $y \in H^{q-p}(\mathcal{P}^*(M^{2n}), d_l^+)$. By definition of the connecting homomorphism we have $\delta x = [\hat{d}_l^- y] = x$. This completes the proof of Lemma 5.4. \square

Clearly Lemma 5.4 implies the first assertion of Theorem 5.2.

Now let us consider the case $p = 0$, $q \leq n-1$. Then $E_{l,1}^{-1,q} = 0$. The previous short exact sequence (5.8) of chain complexes is now replaced by the new one, where the cohomology groups on the line containing $E_{l,1}^{-1,q}$ left and right to $E_{l,1}^{-1,q}$ are zero. Let us write the new short exact sequence explicitly as the following commutative diagram

$$\begin{array}{ccccccc} 0 & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \xrightarrow{\quad} & \\ \downarrow 0 & & \downarrow 0 & & \downarrow 0 & & \\ H_l^q(M^{2n}) & \xrightarrow{[\Pi_{pr}]} & H^q(\mathcal{P}^*(M^{2n}), d_l^+) & \xrightarrow{[d_l^-]} & H_{l-1}^{q-1}(M^{2n}) & \xrightarrow{\quad} & \\ \downarrow \tilde{d}_l=0 & & \downarrow \hat{d}_l^- & & \downarrow \tilde{d}_{l-1}=0 & & \\ H_{l-1}^{q-1}(M^{2n}) & \xrightarrow{[\Pi_{pr}]} & H^{q-1}(\mathcal{P}^*(M^{2n}), d_{l-1}^+) & \xrightarrow{[d_{l-1}^-]} & H_{l-2}^{q-2}(M^{2n}) & \xrightarrow{\quad} & \end{array}$$

The resulting exact sequence of the cohomology groups now are

$$0 \rightarrow H_l^q(M^{2n}) \rightarrow E_{l,2}^{0,q} \rightarrow H_{l-1}^{q-1}(M^{2n}) \xrightarrow{\delta} H_{l-1}^{q-1}(M^{2n}) \dots$$

Since $\delta = Id$, we obtain

$$(5.14) \quad E_{l,2}^{0,q} = H_l^q(M^{2n}),$$

which proves the second assertion of Theorem 5.2.

Next we compute $E_{l,2}^{p,n} = E_{l,1}^{p,n}/d_{l,1}(E_{l,1}^{p-1,n})$ for $0 \leq p \leq n-1$. Since $\omega = d_1\tau$, the map $[L] : H_{l-1}^{n-(p+2)}(M^{2n}) \rightarrow C_l^{n-p}$ sends $[\alpha]$ to $[d_l(\tau \wedge \tilde{\alpha})] = 0 \in C_l^{n-p}$. Thus Proposition 4.5 implies that the following sequence is exact for $0 \leq p \leq n-1$

$$(5.15) \quad 0 \rightarrow C_l^{n-p} \xrightarrow{[L^p]} E_{l+p,1}^{p,n} \xrightarrow{\delta_{p,n}} C_{l-1}^{n-(p+1)} \xrightarrow{[L^{p+1}]} H_{l+p}^{n+p+1}(M^{2n}) \rightarrow 0.$$

Set for $-1 \leq p \leq n-1$

$$(5.16) \quad T_{l-1}^{n-(p+1)} := \ker[L^{p+1}] : C_{l-1}^{n-(p+1)} \rightarrow H_{l+p}^{n+p+1}(M^{2n}).$$

Then we obtain from the exact sequence (5.15) the following short exact sequence

$$(5.17) \quad 0 \rightarrow C_l^{n-p} \xrightarrow{[L^p]} E_{l+p,1}^{p,n} \xrightarrow{\delta_{p,n}} T_{l-1}^{n-(p+1)} \rightarrow 0.$$

Using the isomorphism $E_{l+p,1}^{p,n} = \mathcal{P}^{n-p}(M^{2n})/d_l^+(\mathcal{P}^{n-p-1}(M^{2n}))$ and the formulas (4.18) and (4.21) describing $[L^p]$ and $\delta_{p,n}$ in the exact sequence (4.17) of Proposition 4.5, we rewrite the short exact sequence (5.17) as follows

$$(5.18) \quad 0 \rightarrow C_l^{n-p} \xrightarrow{[\Pi_{pr}]} \frac{\mathcal{P}^{n-p}(M^{2n})}{d_l^+(\mathcal{P}^{n-p-1}(M^{2n}))} \xrightarrow{[d_l^-]} T_{l-1}^{n-(p+1)} \rightarrow 0.$$

Recall that the map $[\Pi_{pr}]$ is already defined in section 4. It is the quotient map of the map $\Pi_{pr} : (\ker d_l^- \cap \Omega^{n-p}(M^{2n})) \rightarrow \mathcal{P}^{n-p}(M^{2n})$, see the explanation of (4.18).

Next recall that the map $[d_l^-]$ is the quotient map of the map $d_l^- : \mathcal{P}^{n-p}(M^{2n}) \rightarrow \mathcal{P}^{n-p-1}(M^{2n}) \cap \ker d_{l-1}^-$, see the explanation of (4.21). (We now explain why this map is also well-defined in (5.18). First we have $d_l^-(d_l^+ \gamma) = d_{l-1}^+ d_l^- \gamma = d_{l-1}^-(d_l^- \gamma) = 0 \in C_{l-1}^{n-(p+1)}$. Furthermore for $\alpha \in \mathcal{P}^{n-p}(M^{2n})$

$$L^{p+1} d_l^-(\alpha) = L^p(L d_l^- \alpha) = L^p d_l \alpha = d_{l+p} L^p \alpha = 0 \in H_{l+p}^{n+p+1}(M^{2n}).$$

Hence

$$[d_l^-] \left(\frac{\mathcal{P}^{n-p}(M^{2n})}{d_l^+(\mathcal{P}^{n-p-1}(M^{2n}))} \right) \subset T_{l-1}^{n-(p+1)} = \ker[L^{p+1}].$$

Thus $[d_l^-]$ is well-defined.)

Note that the differential $d_{l+p,1} : E_{l+p,1}^{p,n} \rightarrow E_{l+p,1}^{p+1,n}$ induces the following boundary operator

$$(5.19) \quad \hat{d}_l^- : \frac{\mathcal{P}^{n-p}(M^{2n})}{d_l^+(\mathcal{P}^{n-p-1}(M^{2n}))} \rightarrow \frac{\mathcal{P}^{n-p-1}(M^{2n})}{d_{l-1}^+(\mathcal{P}^{n-p-2}(M^{2n}))}, [\tilde{\alpha}] \mapsto [d_l^- \tilde{\alpha}],$$

for $\tilde{\alpha} \in \mathcal{P}^{n-p}(M^{2n})$. The map \hat{d}_l^- is well-defined, since by (3.17) $d_l^- d_l^+ \alpha = d_{l-1}^+ d_l^- \alpha$ for $\alpha \in \mathcal{P}^{n-p-1}(M^{2n})$.

Lemma 5.5. *The short exact sequence (5.18) generates a short exact sequence of the following chain complexes*

$$(5.20) \quad \begin{array}{ccccccc} 0 & \longrightarrow & C_l^{n-p} & \xrightarrow{[\Pi_{pr}]} & \frac{\mathcal{P}^{n-p}(M^{2n})}{d_l^+(\mathcal{P}^{n-p-1}(M^{2n}))} & \xrightarrow{[d_l^-]} & T_{l-1}^{n-(p+1)} \longrightarrow 0 \\ & & \downarrow \tilde{d}_l := 0 & & \downarrow \hat{d}_l^- & & \downarrow \bar{d}_{l-1} := 0 \\ 0 & \longrightarrow & C_l^{n-(p+1)} & \xrightarrow{[\Pi_{pr}]} & \frac{\mathcal{P}^{n-(p+1)}(M^{2n})}{d_l^+(\mathcal{P}^{n-(p+2)}(M^{2n}))} & \xrightarrow{[d_l^-]} & T_{l-1}^{n-(p+2)} \longrightarrow 0 \end{array}$$

Proof of Lemma 5.5. It suffices to show that

$$(5.21) \quad \hat{d}_l^- [\Pi_{pr}] = \tilde{d}_l \Pi_{pr} = 0,$$

$$(5.22) \quad [d_{l-1}^-] \hat{d}_l^- = \bar{d}_{l-1} [d_l^-] = 0.$$

Let $\alpha \in C_l^{n-p}$ and $\tilde{\alpha} \in \Omega^{n-p}(M^{2n})$ its representative. Let $\tilde{\alpha} = \tilde{\alpha}_{pr} + L\tilde{\beta}_{pr} + L^2\gamma$ be the Lefschetz decomposition of $\tilde{\alpha}$. Using $d_l\tilde{\alpha} = d_l^+\tilde{\alpha}$ we obtain $Ld_l^-\tilde{\alpha}_{pr} = Ld_{l-1}^+\tilde{\beta}_{pr} + L^2(d_{l-1}^-\tilde{\beta}_{pr} + d_{l-2}\gamma)$. Hence

$$(5.23) \quad \hat{d}_l^- [\Pi_{pr}]\alpha = [d_l^-\tilde{\alpha}_{pr}] = [d_{l-1}^+\tilde{\beta}_{pr}] = 0 \in \frac{\mathcal{P}^{n-p-1}(M^{2n})}{d_{l-1}^+(\mathcal{P}^{n-p-2}(M^{2n}))}.$$

Let $\beta \in (\mathcal{P}^{n-p}(M^{2n}))/d_l^+(\mathcal{P}^{n-p-1}(M^{2n}))$ and $\tilde{\beta} \in \mathcal{P}^{n-p}(M^{2n})$ its representative. Then

$$(5.24) \quad [d_{l-1}^-] \hat{d}_l^- [\beta] = [d_{l-1}^- d_l^- \tilde{\beta}] = 0.$$

Clearly (5.21) and (5.22) follow from (5.23) and (5.24). This completes the proof of Lemma 5.5. \square

The short exact sequence (5.20) in Lemma 5.5 generates the following associated long exact sequence of the cohomology groups

$$(5.25) \quad E_{l+p,2}^{p-1,n} \rightarrow T_l^{n-p} \xrightarrow{\delta} C_l^{n-p} \rightarrow E_{l+p,2}^{p,n} \rightarrow T_{l-1}^{n-(p+1)} \xrightarrow{\delta} C_{l-1}^{n-(p+1)} \rightarrow,$$

where δ is the connecting homomorphism.

Lemma 5.6. *We have $\delta(x) = x$ for all $x \in T_{l-1}^{n-(p+1)}$ and for all $0 \leq p \leq n-1$.*

Proof of Lemma 5.6. Let $x \in T_{l-1}^{n-(p+1)}$. Using (5.20) we write $x = [d_l^-]y, y \in (\mathcal{P}^{n-p}(M^{2n}))/d_l^+(\mathcal{P}^{n-p-1}(M^{2n}))$. By definition of the connecting homomorphism we have $\delta x = [\hat{d}_l^- y] = x$. This completes the proof of Lemma 5.6. \square

Now let us complete the proof of Theorem 5.2. From Lemma 5.6 and the long exact sequence (5.25) we obtain $E_{l+p,2}^{p,n} = C_l^{n-p}/T_l^{n-p}$. Taking into account (5.16) which defines T_l^{n-p} to be the kernel of the surjective homomorphism $[L^p] : C_l^{n-p} \rightarrow H_{l+p}^{n+p}(M^{2n})$, we conclude that

$$(5.26) \quad E_{l+p,2}^{p,n} = H_{l+p}^{n+p}(M^{2n}) \text{ for } 0 \leq p \leq n-1.$$

Next, by Lemma 4.1 $E_{l,1}^{n,n} = C^\infty(M^{2n})$. Since $d_{l,1}(E_{l,1}^{n,n}) = 0$, using (5.19) we get

$$(5.27) \quad E_{l+n,2}^{n,n} = \frac{C^\infty(M^{2n})}{d_l^-(\mathcal{P}^1(M^{2n}))} = H^0(\mathcal{P}^*(M^{2n}), d_l^-).$$

By Proposition 3.10, d_l^- is proportional to $(d_l)_\omega^*$. Applying the symplectic star operator we get from (5.27)

$$(5.28) \quad E_{l+n,2}^{n,n} = H^0(\mathcal{P}^*(M^{2n}), d_l^-) = H_0(\mathcal{P}^*(M^{2n}), (d_l)_\omega^*) = H_{n+l}^{2n}(M^{2n}).$$

Clearly the third assertion of Theorem 5.2 follows from (5.26) and (5.28). The last assertion of Theorem 5.2 follows immediately. This completes the proof of Theorem 5.2. \square

From the exact sequence (5.5) we obtain immediately the following

Corollary 5.7. *Assume that $\omega = d_1\tau$. For $0 \leq p \leq q \leq n-1$ we have*

$$(5.29) \quad E_{l+p,1}^{p,q} = H_l^{q-p}(M^{2n}) \oplus H_{l-1}^{q-p-1}(M^{2n}).$$

Theorem 5.2 can be generalized as follows. Assume that $\omega^p = d_T\rho$ for some $\rho \in \Omega^{2p-1}(M^{2n})$, in particular $d_T\omega^p = 0$. Clearly $d_p(\omega^p) = 0 = d_T(\omega^p)$ implies that $T = p$, since $L^p : \Omega^1(M^{2n}) \rightarrow \Omega^{1+2p}(M^{2n})$ is injective. Furthermore we have $\omega^{k+T} = d_{k+T}(\rho \wedge \omega)$ for all $t \geq 0$. Thus there exists a minimal number T such that $\omega^T = d_T\rho$ for some $\rho \in \Omega^{2T-1}(M^{2n})$.

Theorem 5.8. (cf. [7, Theorem 3]) *Assume that $\omega^T = d_T\rho$ and $T \geq 2$. Then the spectral sequence $(E_{l,r}^{p,q}, d_{l,r})$ stabilizes at the term $E_{l,T+1}^{*,*}$.*

Proof. Our proof of Theorem 5.8 exploits the construction of the exact couple associated with a filtered complex. We use many ideas from [8]. The main idea is to find a short exact sequence, whose middle term is $E_{l,T}^{*,*}$, and moreover, this short exact sequence is induced from the trivial action of the operator L^T on (a part of) complexes entering in the derived exact couples (cf. with the proof of Theorem 5.2). The condition $T \geq 2$ is necessary for Lemma 5.9 below.

Let us begin with recalling the construction of the derived exact couple associated with a filtered complex $(F^p K_l^*, d_l)$ following [23, p.37-43]. We associate with a filtration $(F^p K_l^*, d_l)$ the following exact couple

$$(5.30) \quad \begin{array}{ccc} D_l^{p+1,*} & \xrightarrow{i} & D_l^{p,*} \\ & \searrow \delta & \downarrow j \\ & & E_{l,1}^{p,*} \end{array}$$

where $D_l^{p,q} := H_l^{p+q}(F^p K_l^*)$, and

$$\rightarrow D_l^{p+1,q-1} \xrightarrow{i} D_l^{p,q} \xrightarrow{j} E_{l,1}^{p,q} \xrightarrow{\delta} D_l^{p+1,q} \xrightarrow{i} D_{l,1}^{p,q+1} \xrightarrow{j} E_{l,1}^{p,q+1} \rightarrow$$

is the long exact sequence of cohomology groups associated with the following short exact sequence of chain complexes

$$(5.31) \quad 0 \rightarrow (F^{p+1} K_l^{p+q}, d_l) \xrightarrow{\tilde{i}} (F^p K_l^{p+q}, d_l) \xrightarrow{\tilde{j}} (E_{l,0}^{p,q}, d_{l,0}) \rightarrow 0.$$

The differential $d_{l,1} : E_{l,1}^{p,q} \rightarrow E_{l,1}^{p+1,q}$, defined in (5.1), satisfies the following relation: $d_{l,1} = j \circ \delta$. We refer the reader to [23] for a comprehensive exposition on the relation between the spectral sequence of a filtration and its associated exact couple.

Set $(D_l^{*,*})^0 := D_l^{*,*}$. We define the t -th derived exact couple of the exact couple (5.30), $t \geq 1$,

$$(5.32) \quad (D_l^{p+1,q-1})^{(t)} \xrightarrow{i^{(t)}} (D_l^{p-t,q+t})^{(t)} \xrightarrow{j^{(t)}} E_{l,t+1}^{p,q} \xrightarrow{\delta^{(t)}} (D_l^{p+1,q})^{(t)}$$

inductively as follows [23, p. 38].

$$(5.33) \quad (D_l^{p,q})^{(t)} := i(D_l^{p+1,q-1})^{(t-1)} \subset D_l^{p,q},$$

$$(5.34) \quad i^{(t)}(i^t x) := i(i^t x) \text{ for } i^t x \in (D_l^{p,q})^{(t)},$$

$$(5.35) \quad E_{l,t+1}^{p,q} := \frac{\ker d_{l,t} \cap E_{l,t}^{p,q}}{d_{l,t-1}(E_{l,t}^{p-t+1,q+t-2})},$$

$$(5.36) \quad j^{(t)}(i^t x) := [j^{(t-1)} \circ (i^{(t-1)} x)],$$

$$(5.37) \quad \delta^{(t)}([e]) := \delta^{(t-1)}(e) \in i(D_l^{p,q})^{(t-1)},$$

$$(5.38) \quad d_{l,t+1} := j^{(t)} \circ \delta^{(t)}.$$

Next we consider the following commutative diagram

$$(5.39) \quad \begin{array}{ccc} D_l^{0,q-p} & \xrightarrow{\bar{L}} & D_{l+1}^{0,q-p+2} \\ \downarrow \bar{L}^p & & \downarrow \bar{L}^{p-1} \\ D_{l+p}^{p,q} & \xrightarrow{i} & D_{l+p}^{p-1,q+1} \end{array}$$

where \bar{L}^p is induced by the linear operator $L^p : \Omega^{q-p}(M^{2n}) \rightarrow \Omega^{q+p}(M^{2n})$.

The diagram (5.39) leads us to consider the following diagram

$$(5.40) \quad \begin{array}{ccc} (D_l^{0,q-p})^{(t)} & \xrightarrow{\bar{L}^{(t)}} & (D_{l+1}^{0,q-p+2})^{(t)} \\ \downarrow (\bar{L}^p) & & \downarrow (\bar{L}^{p-1}) \\ (D_{l+p}^{p,q})^{(t)} & \xrightarrow{i^{(t)}} & (D_{l+p}^{p-1,q+1})^{(t)} \end{array}$$

where $\bar{L}^{(t)}$ (resp. (\bar{L}^p)) is the restriction of \bar{L} (resp. \bar{L}^p) to $(D_{l+p}^{p,q})^{(t)}$.

Lemma 5.9. *For $t \geq 1$ and $p \geq 1$, $T \geq 2$, the following statements hold.*

1. *The diagram (5.40) is commutative.*
2. *(\bar{L}^p) is an isomorphism.*
3. *$\text{Im}(i^{(t)}) = \text{Im}((\bar{L}^{p-1}))$.*
4. *$(\bar{L}^{(T-1)})(D_l^{0,q-p})^{(T-1)} = 0$, if $d_T \omega^T = 0$.*

Proof. 1. The commutativity of (5.40) is an immediate consequence of the commutativity of the diagram (5.39).

2. We prove the second assertion of Lemma 5.9 by induction, beginning with $t = 1$. Let $x = i(\alpha) \in (D_{l+p}^{p,q})^{(1)} = i(D_{l+p}^{p+1,q-1}) = i(H_{l+p}^{p+q}(F^{p+1}K^*)) \subset D_{l+p}^{p,q}$. Then there is an element $\alpha' \in \Omega^{q-p-2}(M^{2n})$ such that $[L^{p+1}\alpha'] = \alpha \in D_{l+p}^{p+1,q-1}$, or equivalently $d_{l+p}(L^{p+1}\alpha') = 0$. Hence $L^{p+1}(d_{l-1}(\alpha')) = 0$.

Since $d_{l-1}\alpha' \in \Omega^{q-p-1}(M^{2n})$, $L^{p+1}(d_{l-1}(\alpha')) = 0$ implies that $d_{l-1}\alpha' = 0$, so $\alpha' \in H_{l-1}^{q-p-2}(M^{2n})$. Hence $x = \bar{L}^p(L\alpha')$, and $L\alpha' = i(\alpha') \in (D_l^{0,q-p})^{(1)}$. This proves that the linear map (\bar{L}^p) is surjective for $t = 1$. Furthermore, the map (\bar{L}^p) is injective for $t = 1$, since $L^p : \Omega^{q-p}M^{2n} \rightarrow \Omega^{q+p}M^{2n}$ is injective, and $L^p d_l = d_{l+p}L^p$. This proves Lemma 5.9.2 for $t = 1$.

Now assume that Lemma 5.9.2 has been proved for $t = k$. Since $(D_l^{0,q-p})^{(k+1)}$ is a subset of $(D_l^{0,q-p})^{(k)}$ the injectivity of (\bar{L}^p) follows from the inductive statement. The surjectivity of (\bar{L}^p) also follows from the commutativity of the diagram (5.40), which implies that (\bar{L}^{p-1}) maps the image $i((D_l^{0,q-p})^{(k)})$ onto the set $i^{(k)}(D_{l+p}^{p,q})^{(k)} = (D_{l+p}^{p-1,q+1})^{(k+1)}$. This proves Lemma 5.9.2 for all $t \geq 2$.

3. Clearly Lemma 5.9.3 is a consequence of Lemma 5.9.2 and the commutativity of the diagram (5.40).

4. Let us compute $\bar{L}^{(T-1)}\beta$ for $\beta \in (D_l^{0,q-p-2})^{(T-1)}$, where $T \geq 2$. By definition $\bar{L}^{(T-1)}\beta = \bar{L}(i^{(T-1)}[\tilde{\beta}])$, where $\tilde{\beta} = L^{T-1}\hat{\beta} \in \Omega^{q-p-2}(M^{2n})$ for some $\hat{\beta} \in \Omega^{q-p-2T}(M^{2n})$, and $[\tilde{\beta}] \in D_l^{0,q-p}$, in particular we have $d_l\tilde{\beta} = 0$. Thus $L^{T-1}d_{l-T-1}\tilde{\beta} = 0$. Since $L^{T-1} : \Omega^{q-p-2T+2}(M^{2n}) \rightarrow \Omega^{q-p}(M^{2n})$ is injective we get $d_{l-T+1}\tilde{\beta} = 0$. Now we have

$$(5.41) \quad \bar{L}^{(T-1)}\beta = \bar{L}(i^{(T-1)}\tilde{\beta}) = i^{(T-1)}([L^T\hat{\beta}]) \in (D_{l+p}^{p-1,q+1})^{(T)},$$

by Lemma 5.9.1.

Note that

$$(5.42) \quad [L^T\hat{\beta}] = [d_T\rho \wedge \hat{\beta}] = [d_{T+(l-T+1)}(\rho \wedge \hat{\beta})] = 0 \in D_{l+1}^{0,q-p+2}.$$

Clearly (5.41) and (5.42) imply the last assertion of Lemma 5.9. \square

Lemma 5.9.4 and the $(T-1)$ th-derived exact couple yield the following short exact sequence

$$(5.43) \quad 0 \rightarrow (D_l^{p-(T-1),q+(T-1)})^{(T-1)} \xrightarrow{j^{(T-1)}} E_{l,T}^{p,q} \xrightarrow{\delta^{(T-1)}} (D_l^{p+1,q})^{(T-1)} \rightarrow 0.$$

Lemma 5.10. *The short exact sequence (5.43) generates a short exact sequence of the following chain complexes*

$$\begin{array}{ccccccc} 0 & \longrightarrow & (D_l^{p-(T-1),q+(T-1)})^{(T-1)} & \xrightarrow{j^{(T-1)}} & E_{l,T}^{p,q} & \xrightarrow{\delta^{(T-1)}} & (D_l^{p+1,q})^{(T-1)} \longrightarrow 0 \\ & & \downarrow \tilde{d}_{l,T=0} & & \downarrow d_{l,T} & & \downarrow \tilde{d}_{l,T=0} \\ 0 & \longrightarrow & (D_l^{p+1,q})^{(T-1)} & \xrightarrow{j^{(T-1)}} & E_{l,T}^{p+T,q-T+1} & \xrightarrow{\delta^{(T-1)}} & (D_l^{p+T+1,q-T+1})^{(T-1)} \longrightarrow 0 \end{array}$$

Proof. It suffices to show that

$$(5.44) \quad d_{l,T}j^{(T-1)} = 0,$$

$$(5.45) \quad \delta^{(T-1)}d_{l,T} = 0.$$

The equality $d_{l,T}j^{(T-1)} = 0$ holds, since $d_{l,T}$ is a quotient map of the linear operator d_l acting on $\Omega^*(M^{2n})$, and $j^{(T-1)}$ associates a cycle in $D_l^{p-(T-1),q+(T-1)} \subset H_l^{p+q}(M^{2n})$ to its class in $E_{l,T}^{p,q}$.

The equality $\delta^{(T-1)}d_{l,T} = 0$ holds, since $\delta^{(T-1)}d_{l,T} = \delta^{(T-1)}j^{(T-1)}\delta^{(T-1)} = 0$. This completes the proof of Lemma 5.10. \square

Let us continue the proof of Theorem 5.8. From Lemma 5.10 we obtain the following long exact sequence of the associated cohomology groups

$$(D_l^{p-(T-1),q+(T-1)})^{(T-1)} \xrightarrow{j^*} E_{l,T+1}^{p,q} \xrightarrow{\delta^*} (D_l^{p+1,q})^{(T-1)} \xrightarrow{\partial} (D_l^{p+1,q})^{(T-1)}$$

Lemma 5.11. *For $0 \leq p \leq q \leq n$ the connecting homomorphism $\partial : (D_l^{p+1,q})^{(T-1)} \rightarrow (D_l^{p+1,q})^{(T-1)}$ in (5.46) is equal to the identity.*

Proof. By (5.43) for any $x \in (D_l^{p+1,q})^{(T-1)}$ there exists $e \in E_{l,T}^{p,q}$ such that $y = \delta^{(T-1)}(e)$. Since $d_{l,T}^{p,q}(e) \in \ker \delta^{(T-1)}$ there exists an element $y \in (D_l^{p+1,q})^{(T-1)}$ such that $j^{(T-1)}(y) = d_{l,T}^{p,q}(e) = j^{(T-1)}\delta^{(T-1)}(e)$. Since $j^{(T-1)}$ is injective, $y = \delta^{(T-1)}(e)$. By definition $\partial(x) = y = \delta^{(T-1)}(e)$. It follows that $\partial(\delta^{(T-1)}e) = \delta^{(T-1)}e$. This completes the proof of Lemma 5.11. \square

Corollary 5.12. *For $p \geq T$ we have $E_{l,T+1}^{p,q} = 0$.*

Proof. For $p \geq T$ Lemma 5.11 yields the following exact sequence

$$(D_l^{*,*})^{(T-1)} \xrightarrow{Id} (D_l^{*,*})^{(T-1)} \xrightarrow{j^*} E_{l,T+1}^{p,q} \xrightarrow{\delta^*} (D_l^{*,*})^{(T-1)} \xrightarrow{Id} (D_l^{*,*})^{(T-1)}$$

which implies Corollary 5.12 immediately. \square

It follows from Corollary 5.12 that $d_{l,T+1} : E_{l,T+1}^{p,q} \rightarrow E_{l,T+1}^{p+T+1,q-T} = 0$ for all $p \geq 0$. This completes the proof of Theorem 5.8. \square

We end this section with presenting a proof of the following stabilization theorem.

Theorem 5.13. *(cf. [7, Theorem 4]) Assume that (M^{2n}, ω, θ) is a compact connected globally conformally symplectic manifold. Then the spectral sequence $(E_{l,k}^{p,q}, d_{l,k})$ stabilizes at the $E_{l,2}^{*,*}$ -term.*

Proof. By Theorem 4.6 it suffices to prove Theorem 5.13 for the case of a symplectic manifold (M^{2n}, ω) , i.e. $\theta = 0$. The proof we present here uses many ideas in the proof of Theorem 2 in Di Pietro's Ph.D. Thesis [8] stated for connected compact symplectic manifolds.

By Lemma 4.1 $E_{l,1}^{p,q} = 0 = E_{l,k}^{p,q}$ if $q < p$ or $q > n$ for all $k \geq 1$. Thus it suffices to examine the terms $E_{l,k}^{p,p}$, $E_{l,k}^{p,p+r}$, for $0 \leq p \leq n-r$, $r \geq 1$, $k \geq 2$.

Lemma 5.14. *Assume that (M^{2n}, ω, θ) is a compact l.c.s. manifold, and $0 \leq p \leq n$.*

1. If (M^{2n}, ω, θ) is a globally conformally symplectic manifold, then $E_{l,k}^{p,p} = \mathbb{R}$ for all l and $k \geq 2$. Moreover $E_{l,k}^{p,p}$ is generated by the p -th power of the symplectic form ω .
2. If (M^{2n}, ω, θ) is not conformally equivalent to a symplectic manifold, then $E_{l,k}^{p,p} = 0$ for all $p \neq l$ and for all $k \geq 1$.

Proof of Lemma 5.14. By (4.16) if $0 \leq p \leq n-1$ then

$$(5.47) \quad E_{l,1}^{p,p} = H_{l-p}^0(M^{2n}).$$

By (4.8) we obtain

$$(5.48) \quad E_{l,1}^{n,n} = C_{l-n}^0 = C^\infty(M^{2n}).$$

We get from (5.48) and (5.2)

$$(5.49) \quad E_{l,2}^{n,n} = C^\infty(M^{2n})/d_{l,1}(E_{l,1}^{n-1,n}) = C^\infty(M^{2n})/d_{l-n}^-(\mathcal{P}^1(M^{2n})) = H^0(\mathcal{P}^*(M^{2n}), d_{l-n}^-),$$

By Corollary 3.14

$$(5.50) \quad H_0(\mathcal{P}^*(M^{2n}), d_l^-) = H_0(\mathcal{P}^*(M^{2n}), (d_{l-n})_\omega^*) = H^{2n}(\Omega^*(M^{2n}), d_l).$$

Note that $d_{l,k}(E_{l,k}^{n,n}) = 0$ and $\text{Im } d_{l,k} \cap E_{l,k}^{n,n} = 0$ for all $k \geq 2$. Using (5.49) we get

$$(5.51) \quad E_{l,k}^{n,n} = E_{l,2}^{n,n} = H_0(\mathcal{P}^*(M^{2n}), d_l^-) \text{ for all } k \geq 2.$$

Combining (5.51), (5.50) with Corollary 3.14 we obtain the assertion of Lemma 5.14 for the cases $p = 0$ or $p = n$.

Now let us consider $E_{k,r}^{p,p}$ with $0 < p < n$. By (4.16) $E_{l,1}^{p,p} = H_{l-p}^0(M^{2n})$.

Let us first assume that M^{2n} is globally conformally symplectic. Using Theorem 4.6 we drop l in the lower index of $d_{l,r}$ and $E_{l,r}^{p,q}$. First we note that $E_0^{p,p}$ is generated by ω^p . Since $E_k^{p,p}$ is a quotient of $E_0^{p,p}$ and $[\omega^p] \in E_k^{p,p}$ for all $k \geq 1$, taking into account $[\omega^n] \neq 0 \in E_k^{n,n}$, we complete the proof of Lemma 5.14.1.

Now let us assume that $[\theta] \neq 0 \in H^1(M^{2n})$. Then Corollary 3.14 asserts that $H_l^0(M^{2n}) = 0$ for all $l \neq p$. It follows that $H_{l,\infty}^{p,p} = 0$ for all $l \neq p$. This complete the proof of Lemma 5.14. \square

Lemma 5.15. *Assume that (M^{2n}, ω) is a connected compact symplectic manifold. Then for $0 \leq p \leq n-2$ and $k \geq 2$ we have $E_k^{p,p+1} = H^1(M^{2n})$. Furthermore $E_k^{n-1,n} = E_2^{n-1,n}$ for all $k \geq 2$.*

Proof of Lemma 5.15. Using (5.1) and (5.2) we note that $d_1 : E_1^{0,1} \rightarrow E_1^{1,1}$ is equivalent to the map $d^- : H^1(\mathcal{P}^*(M^{2n}), d^+) \rightarrow H^0(M^{2n}, \mathbb{R}) = \mathbb{R}$. By (3.23) $H^1(\mathcal{P}^*(M^{2n}), d^+) = H^1(M^{2n})$. Hence

$$(5.52) \quad E_2^{0,1} = H^1(M^{2n}).$$

It follows that, the image $d_k(E_k^{0,1}) = 0$ for all $k \geq 2$. Thus we get from (5.52)

$$(5.53) \quad E_k^{0,1} = H^1(M^{2n}) \text{ for all } k \geq 2.$$

This proves Lemma 5.15 for $E_k^{0,1}$, $k \geq 2$. Since the operator $L^p : \Omega^1(M^{2n}) \rightarrow \Omega^{2p+1}(M^{2n})$ is injective for all $p \leq n-1$, using Lemma 5.14 we get

$$(5.54) \quad E_k^{0,1} \cong E_k^{0,1} \wedge E_k^{p,p} \subset E_k^{p,p+1} \text{ for all } k \geq 2.$$

Note that $E_k^{p,p+1}$ is a quotient of $E_2^{p,p+1}$, which is isomorphic to $E_2^{0,1}$ by Proposition 5.1. Taking into account (5.53) we obtain from (5.54)

$$(5.55) \quad E_k^{p,p+1} \cong E_2^{0,1} = H^1(M^{2n}) \text{ for all } p \leq n-2 \text{ and } k \geq 2.$$

This completes the proof the first assertion of Lemma 5.15. The second assertion of Lemma 5.15 follows from the observation that $d_2(E_2^{n-1,n}) = 0 = \text{Im } d_2 \cap E_{l,2}^{n-1,n}$, and $d_k(E_k^{n-1,n}) = 0 = \text{Im } d_k \cap E_k^{n-1,n}$ for all $k \geq 3$. \square

Lemma 5.16. *Assume that (M^{2n}, ω) is a connected compact symplectic manifold. Then $E_k^{n-2,n} = E_2^{n-2,n}$ for all $k \geq 2$. Furthermore, for $0 \leq p \leq n-3$ and $k \geq 2$ we have*

$$(5.56) \quad E_k^{p,p+2} \cong E_2^{0,2}.$$

Proof. First we note that for $k \geq 2$

$$\begin{aligned} d_k(E_k^{n-2,n}) &= 0, \\ \text{Im } d_k \cap E_k^{n-2,n} &= 0. \end{aligned}$$

Hence

$$(5.57) \quad E_k^{n-2,n} = E_2^{n-2,n} \text{ for all } k \geq 2.$$

This proves the first assertion of Lemma 5.16. Next we observe that $d_2(E_2^{0,2}) = 0$ and $\text{Im } d_2 \cap E_2^{0,2} = 0$. Hence we get

$$(5.58) \quad E_k^{0,2} = E_2^{0,2} \text{ for all } k \geq 2.$$

Now we assume that $0 \leq p \leq n-3$. Since $L^p : \Omega^2(M^{2n}) \rightarrow \Omega^{2+2p}(M^{2n})$ is injective, using Lemma 5.14 we get

$$(5.59) \quad E_k^{0,2} \cong E_k^{0,2} \wedge E_k^{p,p} \subset E_k^{p,p+2}.$$

Since $E_k^{p,p+2}$ is a quotient group of $E_2^{p,p+2}$, which is isomorphic to $E_2^{0,2}$ by Proposition 5.1, using (5.58) and (5.59) we get

$$(5.60) \quad E_k^{p,p+2} \cong E_2^{0,2} \text{ for all } 0 \leq p \leq n-3.$$

This completes the proof of Lemma 5.16. \square

Lemma 5.17. *We have $E_k^{p,p+r} = E_2^{0,r}$ for all $k \geq 2$, $p+r \leq n-1$ and $r \geq 3$. Furthermore $E_k^{n-r,n} = E_2^{n-r,n}$ for all $k \geq 2$ and $r \geq 3$.*

Proof. We prove Lemma 5.17 inductively on r beginning with $r = 3$. For each r we will consider $E_k^{p,p+r}$ with k and p increasing inductively. First we note that

$$(5.61) \quad d_2(E_2^{0,3}) = 0 \in E_2^{2,2},$$

since $E_2^{2,2} = E_k^{2,2}$ for all $k \geq 2$ by Lemma 5.14. From (5.61) we obtain easily

$$(5.62) \quad E_k^{0,3} = E_2^{0,3} \text{ for all } k \geq 2.$$

Now using the injectivity of the map $L^p : \Omega^3(M^{2n}) \rightarrow \Omega^{2p+3}(M^{2n})$ for $p \leq n-4$ and Lemma 5.14, we get from (5.62)

$$(5.63) \quad E_2^{0,3} = E_k^{0,3} = E_k^{0,3} \wedge E_k^{p,p} \subset E_k^{p,p+3}.$$

Since $E_k^{p,p+3}$ is a quotient group of $E_2^{p,p+3} = E_2^{0,3}$, (5.63) implies

$$(5.64) \quad E_k^{p,p+3} = E_2^{0,3} \text{ for all } 0 \leq p \leq n-4, k \geq 2.$$

This proves the first assertion of Lemma 5.17 for $r = 3$. The second assertion of Lemma 5.17 for $r = 3$ follows from the identities $\text{Im}d_k \cap E_k^{n-3,n} = 0$ and $d_k(E_k^{n-3,n}) = 0 \in E_k^{n+k-3,n-k+1}$ if $k \geq 2$, which is a consequence of Lemma 5.14 if $k = 2$.

Repeating this procedure we have for each $n \geq r \geq 3$ the following sequences of identities with $k \geq 2$ and $0 \leq p \leq n-r$. First by induction on r we get

$$(5.65) \quad d_2(E_2^{0,r}) = 0 \in E_2^{2,r-1},$$

since $E_2^{2,r-1} = E_k^{2,r-1}$ for all $k \geq 2$ by the induction step. From (5.65) we obtain immediately

$$(5.66) \quad E_k^{0,r} = E_2^{0,r} \text{ for all } k \geq 2$$

Since the map $L^p : \Omega^r(M^{2n}) \rightarrow \Omega^{2p+r}(M^{2n})$ for $p \leq n-r$ is injective, using Lemma 5.14, we get from (5.66)

$$(5.67) \quad E_2^{0,r} = E_k^{0,r} = E_k^{0,r} \wedge E_k^{p,p} \subset E_k^{p,p+r}.$$

Since $E_k^{p,p+r}$ is a quotient group of $E_2^{p,p+r}$ which is isomorphic to $E_2^{0,r}$ if $p+r \leq n-1$ by Proposition 5.1, (5.67) implies

$$(5.68) \quad E_k^{p,p+r} = E_2^{0,r} \text{ for all } 0 \leq p \leq n-r-1, k \geq 2.$$

Thus we get

$$(5.69) \quad E_k^{0,r} = E_k^{p,p+r} \text{ for all } 0 \leq p \leq n-r-1, k \geq 2.$$

This completes the proof of the first assertion of Lemma 5.17.

The second assertion of Lemma 5.17 for the inductive r follows from the identities $\text{Im}d_k \cap E_k^{n-r,n} = 0$ and $d_k(E_k^{n-r,n}) = 0 \in E_k^{n+k-r,n-k+1}$ if $k \geq 2$, which is a consequence of the induction assumption that $E_k^{n+k-r,n-k+1} = E_2^{n+k-r,n-k+1}$ for $0 \leq k \leq r-1$. \square

Clearly Theorem 5.13 follows from Lemmata 5.14, 5.15, 5.16, 5.17. \square

Remark 5.18. 1. Our stabilization theorem 5.13 gives an answer to Tseng-Yau question on the relation between the group $H^p(M^{2n}, d^+) = E_1^{0,p}(M^{2n})$ for $0 \leq p \leq n-1$ and the cohomology groups $H^*(M^{2n}, \mathbb{R})$.

2. In the next section we show that if (M^{2n}, ω) is a compact Kähler manifold, then the spectral sequence stabilizes already at E_1 -terms, see Theorem 6.2.

6. KÄHLER MANIFOLDS

In this section we prove that the spectral sequence $E_r^{p,q}$ stabilizes at the term $E_1^{*,*}$, if (M^{2n}, J, g) is a compact Kähler manifold and ω is the associated symplectic form (Theorem 6.2).

Let (M^{2n}, J, g, ω) be a compact Kähler manifold. As before, denote by d^* the formal adjoint of d . Since the operator L commutes with the Laplacian $\Delta_d := dd^* + d^*d$ we get the induced Lefschetz decomposition of the space of harmonic forms on M^{2n} , and hence the induced Lefschetz decomposition of $H^*(M^{2n}, \mathbb{R})$. Let us denote by $PH^q(M^{2n}, \mathbb{R})$ the subset of primitive cohomology classes in $H^q(M^{2n}, \mathbb{R})$. Note that each primitive cohomology class $[\alpha] \in PH^q(M^{2n})$ has a representative which is Δ_d -harmonic and primitive.

Proposition 6.1. ([29, Proposition 3.18]) *Assume that (M^{2n}, J, g, ω) is a compact Kähler manifold. For $q \leq n-1$ we have $H_q(\mathcal{P}^*(M^{2n}), (d)_\omega^*) = PH^q(M^{2n}) = H^q(\mathcal{P}^*(M^{2n}), d^+)$.*

Proof. We give here another proof using [4]. Bouche proved that if (M^{2n}, J, g, ω) is a compact Kähler manifold, the coeffective cohomology groups $H^{2n-q}(\mathcal{C}^*M^{2n}, d)$ is isomorphic to the subgroup $H_\omega^{2n-q}(M^{2n}) := \{x \in H^{2n-q}(M^{2n}, \mathbb{R}) \mid Lx = 0\}$ for $0 \leq q \leq n-1$ [4, Proposition 3.1]. Next, using [5, Corollary 2.4.2], or the following formula: $\mathcal{J}d^*\mathcal{J}^{-1} = d_\omega^*$, which is proved in the similar way as (3.41) replacing (3.40) in the proof by (3.38), we observe that the symplectic star operator $*_\omega$ maps $H_\omega^{2n-q}(M^{2n})$ isomorphically onto the group $PH^q(M^{2n})$. As we have noted in Remark 3.15.1, the coeffective cohomology group $H^q(\mathcal{C}^*M^{2n}, d)$ is isomorphic to the primitive homology group $H_q(\mathcal{P}^*M^{2n})$. On the other hand Tseng-Yau proved that the group $H(\mathcal{P}^*M^{2n}, (d)_\omega^-)$ is isomorphic to the group $H_q(\mathcal{P}^*(M^{2n}), d^-)$ [29, Lemma 2.7, part II] as well as to the group $H^{q,+}(M^{2n})$, [29, Proposition 3.5, part II], see also Proposition 3.10 and Proposition 3.13 above. Combining these observations we complete the proof of Proposition 6.1. \square

Theorem 6.2. *Assume that (M^{2n}, J, g, ω) is a compact Kähler manifold. Then the spectral sequence $E_r^{p,q}$ stabilizes at $E_1^{*,*}$.*

Proof. Since (M^{2n}, J, g, ω) is a compact symplectic manifold, by Theorem 5.13 the spectral sequence $E_r^{p,q}$ stabilizes at E_2 -terms. Thus to prove Theorem 6.2 it suffices to show that all the differentials $d_1 : E_1^{p,q} \rightarrow E_1^{p+1,q}$ vanish. By (5.2), if $q \leq n-1$ then $d_1 : E_1^{p,q} \rightarrow E_1^{p+1,q}$ is defined by the image of $d^-\tilde{\alpha}$, $\tilde{\alpha} \in \mathcal{P}^{q-p}(M^{2n})$. In this case it suffices to show that any element

$[\alpha] \in H^{q-p}(\mathcal{P}^*(M^{2n}), d^+)$ has a representative $\alpha \in \mathcal{P}^{q-p}(M^{2n})$ such that $d^-\alpha = 0$. By the Hodge theory for d^+ there is a harmonic representative α of $[\alpha]$ such that $(d^+)^*\alpha = 0$. Lemma 3.12 implies that for such harmonic form α we have $d^-\alpha = 0$. This implies $d_1(E_1^{p,q}) = 0$ for $q \leq n-1$. It remains to consider the image $d_1(E_1^{p,n})$. By (5.19) it suffices to show that any element $[\alpha] \in E_1^{p,n}$ has a representative $\alpha \in \mathcal{P}^{n-p}(M^{2n})$ such that $d^-(\alpha) = 0$. Using the Hodge theory for d^+ and (5.19) we choose α to be the harmonic form. By Lemma 3.12 $d^-(\alpha) = 0$. This completes the proof of Theorem 6.2. \square

7. EXAMPLES

In this section we consider two simple examples of compact l.c.s. manifolds and their primitive cohomologies. The first example is a nilmanifold of Heisenberg type [27], the second example is a 4-dimensional solvmanifold described in [1], [2], [26], [15]. We calculate the primitive cohomology of these examples (Propositions 7.1, 7.2). We study some properties of primitive cohomology groups of a l.c.s. manifold, which is a mapping torus of a co-orientation preserving contactomorphism (Proposition 7.4). We show that the 4-dimensional solvmanifold is a mapping torus of a coorientation preserving contactomorphism of a connected contact 3-manifold, which is not isotopic to the identity (Theorem 7.6).

Let $H(n)$ denote the $(2n+1)$ -dimensional Heisenberg Lie group and Γ its lattice. It is well-known that the nilmanifold $N^{2n+2} := (H(n)/\Gamma) \times S^1$ has a canonical l.c.s. form Ω , which we now describe following [27]. Note that the Lie algebra $\mathfrak{h}(n) \oplus \mathbb{R}$ of $H(n) \times \mathbb{R}$ is given by $\langle X_i, Y_i, Z, A : [X_i, Y_i] = Z \rangle_{\mathbb{R}}$. We denote by x_i, y_i, z, α the dual basis. Clearly $d\alpha = 0$ and $d\Omega = \alpha \wedge \Omega$. Here we use the same notations for the extension of $X_i, Y_i, Z, A, x_i, y_i, z, \alpha, \Omega$ to the right-invariants vector fields or differential forms on $H(n) \times \mathbb{R}$, as well as for the descending vector fields or differential forms on N^{2n+2} .

Proposition 7.1. *Let $(N^{2n+2}, \Omega, \alpha)$ be the l.c.s. nilmanifold described above. All the Lichnerowicz-Novikov cohomology groups $H^*(\Omega^*(N^{2n+2}), d_{k\alpha})$ vanish, if $k \neq 0$. Consequently for $k \neq 0$ all the groups $E_{k,r}^{p,q}$, $r \geq 1$, of the associated spectral sequences vanish, unless $q = n$ and $r = 1$. The group $E_{k,1}^{p,n}$ is infinite dimensional for all $0 \leq p \leq n$.*

Proof. The first assertion of Proposition 7.1 is a consequence of a result due to Millionshchikov, who proved that the Lichnerowicz-Novikov cohomology groups $H^*(\Omega^*(M), d_\theta)$ of a compact nilmanifold M always vanish unless θ presents a trivial cohomology class in $H^1(M, \mathbb{R})$ [24, Corollary 4.2]. The second assertion of Proposition 7.1 for $E_{k,1}^{p,q}$ is a consequence of the first assertion, combining with Lemma 4.1 and the exact sequence (4.12). Since $\Omega = d_\alpha(z)$, applying Theorem 5.2 we obtain the second assertion from the first assertion combining with the particular case $r = 1$ proved above. The

third assertion follows from Lemma 4.1 and from the ellipticity of the operators d_k^+ , see the proof of Proposition 3.13. This completes the proof of Proposition 7.1. \square

Now we shall show an example of a l.c.s. 4-manifold $M_{k,n}$, which is an Inoue surface of type S^- , whose primitive cohomologies are non-trivial, and we will explain an implication of this non-triviality. The 4-manifold $M_{n,k}$ has been described in [1], [2], [26], [15]. Here we follow the exposition in [2]. Let G_k be the group of matrices of the form

$$\begin{pmatrix} e^{kz} & 0 & 0 & x \\ 0 & e^{-kz} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $x, y, z \in \mathbb{R}$ and $k \in \mathbb{R}$ such that $e^k + e^{-k} \in \mathbb{Z} \setminus \{2\}$. The group G_k is a connected solvable Lie group with a basis of right invariant 1-forms

$$(7.1) \quad dx - kxdz, \quad dy + kydz, \quad dz.$$

There exists a discrete subgroup $\Gamma_k \subset G_k$ such that $N_k = G_k/\Gamma_k$ is compact. The basis (7.1) descends to a basis of 1-forms α, β, γ on N_k . The forms γ and $\alpha \wedge \beta$ are closed and their cohomology classes generate $H^1(M, \mathbb{R})$ and $H^2(M, \mathbb{R})$ respectively.

Now let $\lambda \in \mathbb{R}$ be a number such that $\lambda[\alpha \wedge \beta] \in H^2(M, \mathbb{Z})$. For given k, n denote by $M_{k,n}$ the total space of the S^1 -principal bundle over N_k with the Chern class $n\lambda[\alpha \wedge \beta]$. Let η be a connection form on $M_{k,n}$, equivalently

$$(7.2) \quad d\eta = n\lambda(\alpha \wedge \beta).$$

For simplicity we will denote the pull back to $M_{k,n}$ by the projection $M_{k,n} \rightarrow N_k$ of a form θ on N_k again by θ . Banyaga showed that $M_{k,n}$ possesses many interesting l.c.s. structures. Here we consider only two l.c.s. forms $d_{-k\gamma}\eta = n\lambda(\alpha \wedge \beta) - k\gamma \wedge \eta$ and $d_{k\gamma}\eta = n\lambda(\alpha \wedge \beta) + k\gamma \wedge \eta$ discovered by Banyaga [2, Remark 2]. Note that $M_{k,n}$ carries no symplectic structure, since $H^2(M_{k,n}, \mathbb{R}) = 0$ [1]. Since $M_{k,n}$ is compact, the Hodge theory applied to $d_{\pm\gamma}$ yields that $H^{2-i}(\Omega^*(M_{n,k}), d_{\pm k\gamma}) = H^{2+i}(\Omega^*(M_{n,k}), d_{\pm k\gamma})$. Since $[\pm k\gamma] \neq 0 \in H^1(M_{n,k}, \mathbb{R})$, the Lichnerowicz deformed differential $d_{\pm k\lambda}$ is not gauge equivalent to the canonical differential d . Hence $H^0(\Omega^*(M^{2n}), d_{\pm k\gamma}) = 0$. Denote by $\mathcal{P}_{\pm}^*(M^{2n})$ the space of primitive forms corresponding to the l.c.s. form $d_{\pm k\gamma}\eta$. Corollary 3.14 yields that $H^0(\mathcal{P}_{\pm}^*(M_{k,n}), d_{\pm k\gamma}^+) = 0$ for all $l \neq 0$, and $H^0(\mathcal{P}_{\pm}^*(M^{2n}), d) = \mathbb{R}$.

Proposition 7.2. 1. $H^1(\Omega^*(M_{k,n}), d_{\pm k\gamma}) = \mathbb{R}$.

2. $H^2(\Omega^*(M_{k,n}), d_{\pm k\gamma}) = \mathbb{R}$.

3. $H^1(\mathcal{P}_{\pm}^*(M_{k,n}), d_{\pm k\gamma}^+) = \mathbb{R}^2$.

Proof. It is known that $M_{k,n}$ is a complete solvmanifold. Indeed, the algebra $\mathfrak{g}_{k,n}$ of the corresponding solvable group possesses the basis (X, Y, Z, T) dual to $(\alpha, \beta, \gamma, \eta)$ with the following properties [1], or see (7.5) and (7.6) below.

$$(7.3) \quad [X, Z] = kX, [X, Y] = -n\lambda T, [Y, Z] = -kY,$$

$$(7.4) \quad [X, T] = [Y, T] = [Z, T] = 0.$$

Using (7.3) and (7.4) we observe that the Lie subalgebras $\langle T \rangle_{\mathbb{R}} \subset \langle T, X \rangle_{\mathbb{R}} \subset \langle T, X, Y \rangle_{\mathbb{R}}$ are ideals of $\mathfrak{g}_{k,n}$, so $M_{k,n}$ is completely solvable. Now we apply the result by Millionshchikov [24, Corollary 4.1, Theorem 4.5], which reduces the computation of the Novikov cohomology groups of a compact complete solvmanifold G/Γ to the computation of the induced Novikov cohomology groups of the Lie algebra \mathfrak{g} of G . For our computation it is useful to rewrite (7.3) and (7.4) in the dual basis of $\mathfrak{g}_{k,n}^*$, or using the explicit formulae for $\alpha, \beta, \gamma, \eta$ given in (7.1), (7.2) above to obtain

$$(7.5) \quad d\alpha = -k\alpha \wedge \gamma, d\beta = k\beta \wedge \gamma,$$

$$(7.6) \quad d\gamma = 0, d\eta = n\lambda(\alpha \wedge \beta).$$

1. Abbreviate $d_{\pm k}\gamma$ as $d_{\pm k}$. Using (7.5) and (7.6) we get

$$(7.7) \quad d_{\pm k}\alpha = (k \pm k)\gamma \wedge \alpha, d_{\pm k}\beta = (-k \pm k)\gamma \wedge \beta,$$

$$(7.8) \quad \gamma = d_{\pm k}(\pm 1/k), d_{\pm k}\eta = n\lambda(\alpha \wedge \beta) \pm k\gamma \wedge \eta.$$

Using (7.7) and (7.8) it is easy to compute that α is a generator of $H^1(\Omega^*(M_{k,n}), d_{-k})$, and β is a generator of $H^1(\Omega^*(M_{k,n}), d_k)$. This proves the first assertion of Proposition 7.2.

2. For computing $H^2(\Omega^*(M_{k,n}), d_{\pm k})$ we use (7.7), (7.8), and the following formulae

$$\begin{aligned} d_{\pm k}(\alpha \wedge \beta) &= \pm k\alpha \wedge \beta \wedge \gamma, \\ d_{-k}(\alpha \wedge \gamma) &= 0, d_k(\beta \wedge \gamma) = 0, \\ d_{\pm k}(\alpha \wedge \eta) &= (-k \pm k)\gamma \wedge \alpha \wedge \eta, d_{\pm k}(\beta \wedge \eta) = (k \pm k)\gamma \wedge \beta \wedge \eta, \\ d_{\pm k}(\gamma \wedge \eta) &= -n\lambda\alpha \wedge \beta \wedge \gamma. \end{aligned}$$

It is easy to see that $\alpha \wedge \eta$ is a generator of $H^2(\Omega^*(M_{k,n}), d_{-k})$ and $\beta \wedge \eta$ is a generator of $H^2(\Omega^*(M_{k,n}), d_k)$. This proves the second assertions of Proposition 7.2.

3. The third assertion of Proposition 7.2 is a consequence of the first assertion and Formula(3.24). This completes the proof of Proposition 7.2. \square

In the remaining part of this section we study some properties of primitive cohomology groups of l.c.s. manifolds associated with a co-orientation preserving contactomorphism. We show that the l.c.s. solvmanifold studied before is an example of a l.c.s. manifold associated with a non-trivial contactomorphism.

Let (M^{2n+1}, α) be a co-orientable contact manifold and f be a co-orientation preserving contactomorphism of (M^{2n+1}, α) , i.e. $f^*(\alpha) = e^h \cdot \alpha$ for some

$h \in C^\infty(M^{2n})$. The mapping torus $M_f^{2n+2} = (M \times [0, 1]) / ([x, 0] = [f(x), 1])$ of a contactomorphism f is a fibration over S^1 whose fiber is M^{2n+1} . Let us denote this fibration by $\pi : M_f^{2n+2} \rightarrow S^1$ with $\pi^{-1}(s) = [M, s]$. Let $f_t : M_f^{2n+2} \rightarrow M_f^{2n+2}$ be a 1-parameter family of diffeomorphisms defined by:

$$f_t([x, s]) = [x, s + t \mod 1] \text{ for } t \in \mathbb{R}.$$

In particular $f_1([x, 0]) = [f(x), 0]$. Let us also denote by α the contact 1-form on $[M, 0]$ obtained by identifying M with $[M, 0]$. Let B be the vector field on M_f^{2n+2} defined by $B([x, s]) = (d/dt)|_{t=0} f_t([x, s])$. Since $f^*(\alpha) = e^h \alpha$ the following 1-form $\tilde{\alpha}$

$$(7.9) \quad \tilde{\alpha}(x, t)|_{\pi^{-1}(t)} := e^{-th(x)} f_{-t}^*(\alpha), \quad \tilde{\alpha}(B) = 0.$$

is well-defined on M_f^{2n+2} , moreover

$$f_t^*(\tilde{\alpha})|_{\pi^{-1}(t)} = \tilde{\alpha}|_{\pi^{-1}(0)} \text{ for all } 0 \leq t \leq 1.$$

Set $\theta := \pi^*(dt)$.

Proposition 7.3. (cf. [3, Proposition 3.3.]) 1. Assume that (M^{2n+1}, α) is a compact co-orientable contact manifold and f is a co-orientation preserving contactomorphism. There exists a positive number c_0 such that $(M_f^{2n+2}, \omega_c := d\tilde{\alpha} + c\theta \wedge \tilde{\alpha}, c\theta)$ is a l.c.s. manifold for all $c \geq c_0$.

2. Assume that f preserves the contact 1-form α . Then $(M_f^{2n+2}, \omega := d\tilde{\alpha} + \theta \wedge \tilde{\alpha}, \theta)$ is a l.c.s. manifold.

Proof. 1. Clearly (7.9) implies that $rk d\tilde{\alpha} \geq rk d\alpha = 2n$. Using this we conclude that there exists a positive number c_0 such that $rk d\omega_c = 2n + 2$ for all $c \geq c_0$, since M^{2n+1} is compact. Further, $d(c\theta) = 0$ and $\omega_c = d_{c\theta}(\tilde{\alpha})$. This proves that $(M_f^{2n+2}, \omega_c, c\theta)$ is a l.c.s. manifold.

2. Assume that $f^*(\alpha) = \alpha$. Then $\tilde{\alpha}([x, t])|_{\pi^{-1}(t)} = f_{-t}^*\alpha$. It follows that $rk d\tilde{\alpha} = rk d\alpha = 2n$, and $rk \omega_1 = 2n + 2$. Hence $\omega_1 = \omega$ is a l.c.s. form, taking into account $\omega = d_\theta \alpha$. \square

Proposition 7.4. 1. Suppose that f_0 and f_1 are co-orientation preserving contactomorphisms of a compact co-orientable contact manifold (M^{2n+1}, α) . The l.c.s. manifolds $M_{f_0}^{2n+2}$ and $M_{f_1}^{2n+2}$ are diffeomorphic, if f_0 and f_1 are isotopic. For sufficiently large number c the primitive cohomology groups of $(M_{f_0}^{2n+2}, \omega_c, c\theta)$ and of $(M_{f_1}^{2n+2}, \omega'_c, c\theta)$ are isomorphic.

2. Let θ be the Lee form of the associated l.c.s form on M_f^{2n+1} . If f is isotopic to the identity, the Lichnerowicz cohomology groups $H^*(\Omega^*(M_f^{2n+2}), d_{c\theta})$ are zero, for any $c \neq 0$.

Proof. The first assertion of Proposition 7.4.1 is well-known. The second assertion of Proposition 7.4.1 is a consequence of Theorem 4.6, observing that $\omega_c - \omega'_c = d_{c\theta}(\tilde{\alpha} - \tilde{\alpha}')$.

Finally Proposition 7.4.2 follows from the first assertion, combining with the fact that the l.c.s. manifold $(M^{2n+1} \times S^1, d_\theta \tilde{\alpha}, \theta)$ associated to the identity mapping of the contact manifold (M^{2n+1}, α) has vanishing Lichnerowicz-Novikov groups, taking into account the Künneth formula and the formula $H^*(\Omega^*(S^1), d_{c dt}) = 0$ if $c \neq 0$. This completes the proof of Proposition 7.4. \square

Now we shall show that our l.c.s. manifold $(M_{k,n}, d_{k\gamma}\eta, k\gamma)$ is a mapping torus of a non-trivial co-orientation preserving contactomorphism. First we prove the following

Proposition 7.5. *Assume that γ is a closed 1-form on a compact smooth manifold M . If $[\gamma] \in H^1(M, \mathbb{Z})$ and γ is now-where vanishing, then there is a submersion $f : M \rightarrow S^1$ such that $f^*(dt) = \gamma$, where dt is the canonical 1-form on S^1 .*

Proof. We use Tischler's argument in [28]. Since S^1 is the Eilenberg-MacLane space there exists a map $f_1 : M \rightarrow S^1$ such that $f^*([dt]) = [\gamma]$. Without loss of generality we assume that f is a smooth map. Hence we have $f^*(dt) = \gamma + dh$ for some smooth function h on M . Now we observe that $f_1^*(dt) + dh = (f_1 + \Pi \circ h)(dt)$, where $\Pi : \mathbb{R} \rightarrow S^1$ is the natural projection. Clearly the map $f = f_1 + \Pi \circ h$ is a submersion, since γ is no-where vanishing. This completes the proof of Proposition 7.5. \square

Now we are ready to show the following implication of Proposition 7.2.

Theorem 7.6. *The l.c.s. manifold $(M_{k,n}, d_{k\gamma}\eta, k\gamma)$ is a mapping torus of a coorientation preserving contactomorphism f of a 3-dimensional connected contact manifold. Moreover f is not isotopic to the identity.*

Proof. Since $H^1(M_{k,n}, \mathbb{R}) = \mathbb{R}[1]$, and $d\gamma = 0$, there exists a positive number p such that $p[\gamma]$ is a generator of $H^1(M_{k,n}, \mathbb{Z}) = \mathbb{Z} = \text{Hom}(H_1(M_{k,n}, \mathbb{Z}), \mathbb{Z})$ [9, Chapter VI, 7.22]. Applying Proposition 7.5 we conclude that $M_{k,n}$ is a fibration over S^1 whose fibers are the foliation $\mathcal{F}_1 := \{\gamma = 0\}$, and $f^*(dt) = p \cdot \gamma$, since γ is nowhere vanishing. Denote by $\pi : M_{k,n} \rightarrow S^1$ the corresponding fibration. Note that the restriction of η to each fiber $\pi^{-1}(t)$, $t \in S^1$, is a contact form, since X, Y, T are tangent to the fiber and we have $\eta(T) = 1$, $d\eta(X, Y) \neq 0$.

First we will show that the fiber $F := \pi^{-1}(t)$, $t \in S^1$, is connected. Let us consider the following exact sequence of homotopy groups

$$(7.10) \quad \pi_1(M_{k,n}) \rightarrow \pi_1(S^1) \rightarrow \pi_0(F) \rightarrow 0 = \pi_0(M_{k,n}).$$

To show that $\pi_0(F) = 0$ it suffices to prove that the map $\pi_1(M_{k,n}) \rightarrow \pi_1(S^1)$ is surjective. Since $p[\gamma]$ is a generator of $H^1(M_{k,n}, \mathbb{Z})$ there exists an element $a \in H_1(M_{k,n}, \mathbb{Z})$ such that $\langle p[\gamma], a \rangle = 1$. Since $\langle [dt], \pi_*(a) \rangle = \langle p[\gamma], a \rangle = 1$, it follows that $\pi_* : H_1(M_{k,n}, \mathbb{Z}) \rightarrow H_1(S^1)$ is surjective. Hence $\pi_* : \pi_1(M_{k,n}) \rightarrow \pi_1(S^1)$ is surjective. Hence F is connected.

Now let f_t denote the flow on $M_{k,n}$ generated by the vector field Z . We note that $\mathcal{L}_Z(\gamma) = d(\gamma(Z)) = 0$, so f_t respects fibration π . Next we have $\mathcal{L}_Z(\eta) = Z \lrcorner n\lambda\alpha \wedge \beta + d(\eta(Z)) = 0$. Hence f_t preserves also the contact form on the fiber F . This proves the first assertion.

The second assertion is a consequence of Proposition 7.2 and Proposition 7.4. This completes the proof of Theorem 7.6. \square

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